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GINN AND
COMPANY
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ELEMENTS

OF

PLANE AND SOLID GEOMETRY

BY

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SANDERS' PLANE AND SOLID GEOM.

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PURPOSE AND DISTINCTIVE FEATURES

THIS work has been prepared for the use of classes in high schools, academies, and preparatory schools. Its distinctive features are:—

1. *The omission of parts of demonstrations.*

By this expedient the student is forced to rely more on his own reasoning powers, and is prevented from acquiring the detrimental habit of memorizing the text.

As it is necessary for the beginner in Geometry to learn the *form* of a geometrical demonstration, the demonstrations of the first few propositions are given in full. In the succeeding propositions only the most obvious steps are omitted, the omission in each case being indicated by an interrogation mark (?). In no case is the student expected to originate the *plan* of proof.

2. *The introduction, after each proposition, of exercises bearing directly upon the principle of the proposition.*

As soon as a proposition has been mastered, the student is required to apply its principle in the solution of a series of easy exercises. Hints or suggestions are given to aid the pupil in the solution of the more difficult exercises.

3. *All constructions, such as drawing parallels, erecting perpendiculars, etc., are given before they are required to be used in demonstrations.*

4. *Exercises in Modern Geometry.*

Exercises involving the principles of Modern Geometry are given under their proper propositions. As the omission of these exercises cannot affect the sequence of propositions, they may be disregarded at the discretion of the teacher.

5. *Propositions and converses.*

Whenever possible, the converse of a proposition is given with the proposition itself.

6. *Number of exercises.*

Besides the exercises directly following each proposition, miscellaneous exercises are given at the end of each book. It may be found that there are more exercises given than can be covered by a class in the time allotted to the subject of Geometry; in which case the teacher will have to select from the lists given.

While the exercises have been drawn from many sources, the author has availed himself in particular of the recent entrance examination papers of the best American colleges and scientific schools.

The author wishes to express his obligations to his colleagues in the Cincinnati High Schools for their criticism and encouragement, and especially to Miss Celia Doerner of Hughes High School for valuable suggestions and for her painstaking reading of the proof.

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PLANE GEOMETRY

[The references are to articles]

PLANE GEOMETRY

DEFINITIONS

1. Every material body occupies a limited portion of space. If we conceive the body to be removed, the space that is left, which is identical in form and magnitude with the body, is a geometrical solid.

2. A *geometrical solid,* then, is a limited portion of space. It has three dimensions: length, breadth, and thickness.

3. The boundaries of a solid are *surfaces.* A surface has but two dimensions: length and breadth.

4. The boundaries of a surface are *lines.* A line has length only.

5. The ends of a line are *points.* A point has position, but no magnitude.

6. A *straight line* is one that does not change its direction at any point.

7. A *curved line* changes its direction at every point.

8. A *plane surface* is a surface, such that a straight line joining any two of its points will lie wholly in the surface.

9. Any combination of points, lines, surfaces, or solids, is a *geometrical figure.*

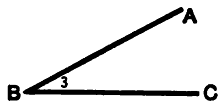
10. A figure formed by points and lines in a plane is a *plane figure.*

11. *Geometry* is the science that treats of the properties, the construction, and the measurement of geometrical figures.

12. *Plane Geometry* treats of plane figures.

13. A *plane angle* is the amount of divergence of two lines that meet. The lines are the *sides* of the angle, and their point of meeting is the *vertex.*

One way to indicate an angle is by the use of three letters. Thus, the angle in the accompanying figure is read *angle ABC* or *angle CBA*, the letter at the vertex being in the middle.

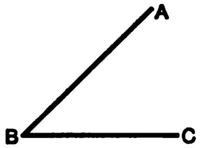


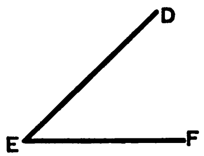
If there is only one angle at the vertex *B*, it may be read *angle B*.

Another way is to place a small figure or letter within the angle near the vertex. The above angle may be read *angle 3.*

The size of an angle in no way depends upon the length of its sides, and is not altered by either increasing or diminishing their length.

14. Two angles are equal if they can be made to coincide. Thus, angles *ABC* and *DEF* are equal, whatever may be the length of each side, if angle *ABC* can be placed upon angle *DEF* so that the vertex *B* shall fall upon vertex *E*, *BC* fall upon *EF*, and *BA* fall upon *ED*.

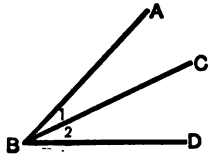




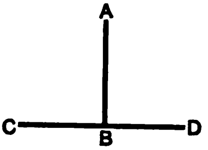
[It should be noticed that angle *ABC* can be made to coincide with angle *DEF* in another way, *i.e.* *ABC* may be *turned over* and then placed upon *DEF*, *BC* falling upon *ED*, and *BA* upon *EF*.]

15. Two angles that have the same vertex and a common side between them are *adjacent angles.*

Angles 1 and 2 are adjacent angles.

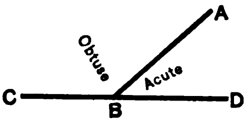


16. If a straight line meets another straight line so as to make the adjacent angles that they form equal to each other, the angles formed are *right angles.* Angles ABC and ABD are right angles. In this case each line is *perpendicular* to the other.

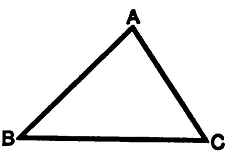


17. An angle that is less than a right angle is *acute,* and one that is greater than a right angle is *obtuse.*

An angle that is not a right angle is *oblique.*



18. A *triangle* is a portion of a plane bounded by three straight lines. The lines are called the *sides of the triangle,* and their angles the *angles of the triangle.*



An *equilateral triangle* has three equal sides.

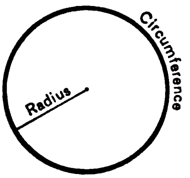
An *isosceles triangle* has two equal sides.

A *scalene triangle* has no two sides equal.

An *equiangular triangle* has three equal angles.

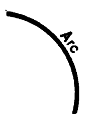
A *right-angled triangle* contains one right angle.

19. A *circle* is a portion of a plane bounded by a curved line, all points of which are equally distant from a point within, called the *center.* The bounding line is called the *circumference.*



20. The distance from the center to any point on the circumference is a *radius.*

21. Any portion of a circumference is an *arc.*



22. A *theorem* is a truth requiring demonstration. The statement of a theorem consists of two parts, the *hypothesis* and the *conclusion.* The *hypothesis* is that part which is assumed to be true; the *conclusion* is that which is to be proved.

23. A *problem* proposes to effect some geometrical construction, such as to draw some particular line, or to construct some required figure.

24. Theorems and problems are called *propositions.*

25. A *corollary* is a truth that may be readily deduced from one or more propositions.

26. A *scholium* is a remark made upon one or more propositions relating to their use, connection, limitation, or extension.

27. An *axiom* is a self-evident truth.

Axioms

1. Things that are equal to the same thing are equal to each other.

2. If equals are added to equals, the sums are equal.

3. If equals are subtracted from equals, the remainders are equal.

4. If equals are multiplied by equals, the products are equal.

5. If equals are divided by equals, the quotients are equal.

6. If equals are added to unequals, the sums are unequal in the same order.

7. If equals are subtracted from unequals, the remainders are unequal in the same order.

8. If unequals are multiplied by positive equals, the products are unequal in the same order.

9. If unequals are divided by positive equals, the quotients are unequal in the same order.

10. If unequals are added to unequals, the greater to the greater, and the less to the less, the sums are unequal in the same order.

11. The whole is greater than any of its parts.

12. The whole is equal to the sum of all its parts.

13. Only one straight line can be drawn joining two points.

[It follows from this axiom that two straight lines can intersect in only one point.]

14. The shortest distance from one point to another is measured on the straight line joining them.

15. Through a point only one line can be drawn parallel to another line.

16. Magnitudes that can be made to coincide with each other are equal.

[This axiom affords the ultimate test of the equality of geometrical magnitudes. It implies that a figure can be taken from its position, without change of form or size, and placed upon another figure for the purpose of comparison.]

Of the foregoing, the first twelve axioms are general in their nature, and the student has probably met with them before in his study of algebra. The last four are strictly geometrical axioms.

28. A *postulate* is a self-evident problem.

Postulates

1. A straight line can be drawn joining two points.

2. A straight line can be prolonged to any length.

3. If two lines are unequal, the length of the smaller can be laid off on the larger.

4. A circumference can be described with any point as a center, and with a radius of any length.

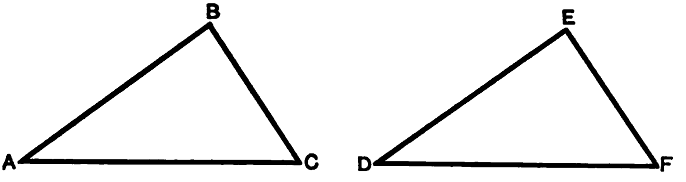
29. SYMBOLS AND ABBREVIATIONS

∠	Angle.
∠s	Angles.
R.A.	Right angle.
R.A.'s.	Right angles.
△	Triangle.
△s	Triangles.
⊙	Circle.
⊙s	Circles.
⊥	Perpendicular.
⊥s	Perpendiculars.
∥	Parallel.
∥s	Parallels.
∴	Therefore.
=	Equals *or* equal.
>	Is (*or* are) greater than.
<	Is (*or* are) less than.
∼	Is (*or* are) measured by.
Prop.	Proposition.
Cor.	Corollary.
Schol.	Scholium.
Q.E.D.	Quod erat demonstrandum, *which was to be proved.*
Q.E.F.	Quod erat faciendum, *which was to be done.*

BOOK I

PROPOSITION I. THEOREM

30. ***If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are equal in all respects.***



Let the △ *ABC* and *DEF* have $AB = DE$, $BC = EF$, and $\angle B = \angle E$.

To Prove the △ *ABC* and *DEF* equal in all respects.

Proof. Place the △ *ABC* upon the △ *DEF* so that ∠ *B* shall coincide with its equal ∠ *E*, *BA* falling upon *ED*, and *BC* upon *EF*.

Since, by hypothesis, $BA = ED$, the vertex *A* will fall upon the vertex *D*.

Since, by hypothesis, $BC = EF$, the vertex *C* will fall upon the vertex *F*.

Since, by Axiom 13, only one straight line can be drawn joining two points, *AC* will coincide with *DF*. ∴ the △ coincide throughout and are equal in all respects. Q.E.D.

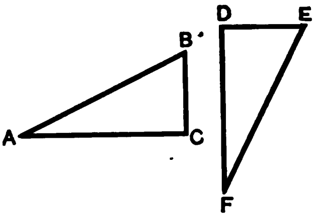
31. Scholium. By showing that the △ coincide, we have not only proved that they are equal in area, but also that $\angle A = \angle D$, $\angle C = \angle F$, and $AC = DF$.

It should be noticed that the sides AC and DF, which have been proved equal, lie opposite respectively to the equal angles B and E.

Also, that the equal angles A and D lie opposite respectively to the equal sides BC and EF, and that the equal angles C and F lie opposite respectively to the equal sides AB and DE.

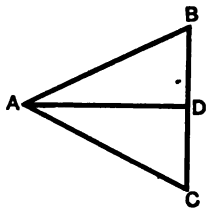
Principle. *In triangles that have been proved equal in all respects, equal sides lie opposite equal angles, and equal angles lie opposite equal sides.*

32. Exercise. Prove Prop. I., using this pair of triangles.



33. Exercise. In the triangle ABC, $AB = AC$, and AD bisects the angle BAC.

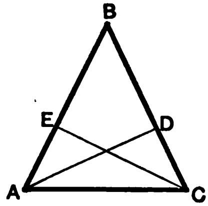
Prove that AD also bisects BC.



Suggestion. Show by § 30 that the △ ABD and ADC are equal in all respects. Then, by the principle of § 31, $BD = DC$.

34. Exercise. ABC is a triangle having $AB = BC$. BE is laid off equal to BD.

Prove $AD = CE$.

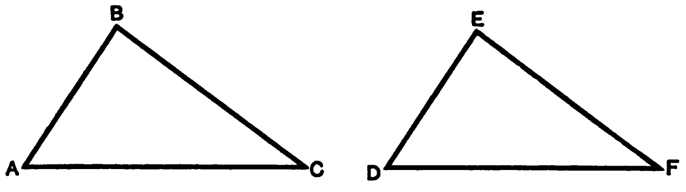


Suggestion. Show that

$$\triangle ABD = \triangle EBC.$$

Proposition II. Theorem

35. ***If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are equal in all respects.***



Let the ⊿s ABC and DEF have $\angle A = \angle D$, $\angle C = \angle F$, and $AC = DF$.

To Prove the ⊿s ABC and DEF equal in all respects.

Proof. Place the $\triangle ABC$ upon the $\triangle DEF$, so that $\angle A$ shall coincide with its equal $\angle D$, AB falling upon DE, and AC falling upon DF.

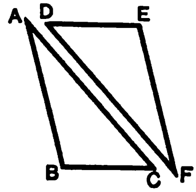
Since, by hypothesis, $AC = DF$, the vertex C will fall upon vertex F.

Since, by hypothesis, $\angle C = \angle F$, the side CB will fall upon FE, and the vertex B will be on FE or its prolongation.

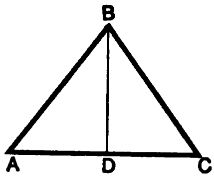
Since AB falls upon DE, the vertex B will be upon DE or its prolongation.

The vertex B, being at the same time on DE and FE, must be at their point of intersection; and since two straight lines have only one point of intersection (Axiom 13), the vertex B must fall at E.

∴ the ⊿s ABC and DEF coincide throughout, and are equal in all respects. Q.E.D.

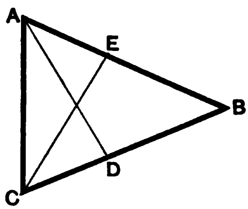


36. Exercise. Prove Prop. II., using this pair of triangles.



37. Exercise. In the $\triangle ABC$, BD bisects $\angle ABC$ and is perpendicular to AC.

Prove that BD bisects AC and that $AB = BC$.



38. Exercise. ABC is a $\triangle$ having $\angle BAC = \angle BCA$. AD bisects $\angle BAC$ and CE bisects $\angle BCA$.

Prove $AD = CE$.

Suggestion. Prove $\triangle$s ADC and AEC equal in all respects by § 35. Then by the Principle of § 31, $AD = EC$.

39. The next proposition is an example of what is called the *indirect proof.*

The reasoning is based on the following Principle: *If the direct consequences of a certain supposition are false, the supposition itself is false.*

To prove a theorem by this plan, the following steps are necessary:

1. The theorem is supposed to be untrue.

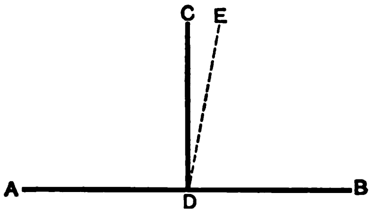
2. The consequences of this supposition are shown to be false.

3. Then, by the above Principle, the supposition (that the theorem is untrue) is false.

4. The theorem is therefore true.

Proposition III. Theorem

40. ***At a given point in a line only one perpendicular can be erected to that line.***



Let CD be ⊥ to AB at the point D.

To Prove CD is the only ⊥ that can be erected to AB at D.

Proof. Suppose a second ⊥, as DE, could be erected to AB at D.

By hypothesis and § 16, $\angle CDA = \angle CDB$.

By supposition and § 16, $\angle EDA = \angle EDB$.

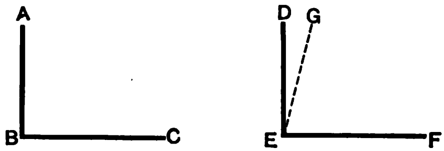
But $\angle EDA > \angle CDA$, and $\angle EDB < \angle CDB$.

∴ $\angle EDA$ cannot equal $\angle EDB$, and DE cannot be ⊥ to AB.

The supposition that a second ⊥ could be erected to AB at D is therefore false, and only one ⊥ can be erected to AB at that point. Q.E.D.

Note. The points and lines of the above figure, and of all figures given in the first five books of this geometry, are understood to be in the same plane. The term "line" is used in this work for "straight line."

41. COROLLARY. *All right angles are equal.*



Let $\angle ABC$ and $\angle DEF$ be 2 R.A.'s.

To Prove $\angle ABC = \angle DEF$.

Proof. Suppose them to be unequal and that $\angle ABC$, when superimposed upon $\angle DEF$, takes the position GEF.

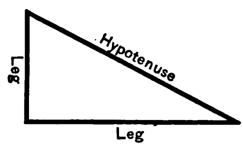
Then at E there would be two perpendiculars to EF, which contradicts § 40.

Therefore the supposition that the right angles ABC and DEF are unequal is false, and they are equal. Q.E.D.

42. SCHOLIUM. The right angle is the unit of measure for angles. An angle is generally expressed in terms of the right angle. Thus, $\angle A = \frac{2}{3}$ R.A., or $\angle B = 1\frac{1}{4}$ R.A., etc.

43. DEFINITIONS. In a *right-angled triangle* the side opposite the right angle is called the *hypotenuse*.

The other two sides are the *legs* of the triangle.



44. EXERCISE. If two *R.A.* ▲ have the legs of one equal respectively to the legs of the other, the ▲ are equal in all respects.

45. EXERCISE. *A* is 40 miles west of *B*. *C* is 30 miles north of *A*, and *D* is 30 miles south of *A*. From *C* to *B* is 50 miles. How far is it from *D* to *B*?

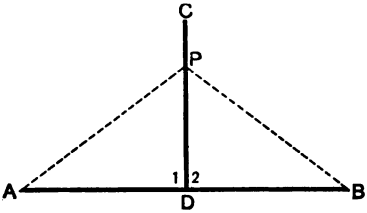
46. EXERCISE. *A* is *m* yards north of *B*. *C* is *n* yards west of *A*, and *D* is *n* yards east of *B*. Prove that the distance from *B* to *C* is the same as the distance from *A* to *D*.

PROPOSITION IV. THEOREM

47. *If a perpendicular is drawn to a line at its middle point,*

I. *Any point on the perpendicular is equally distant from the extremities of the line.*

II. *Any point without the perpendicular is unequally distant from the extremities of the line.*



I. **Let** *CD* be ⊥ to *AB* at its middle point *D*, and *P* be any point on *CD*.

To Prove *P* equally distant from *A* and *B*.

Draw *PA* and *PB*.

[It is required to prove $PA = PB$, for *PA* and *PB* measure the distance from *P* to *A* and *B* respectively.]

Proof. The △ *PAD* and *PBD* have

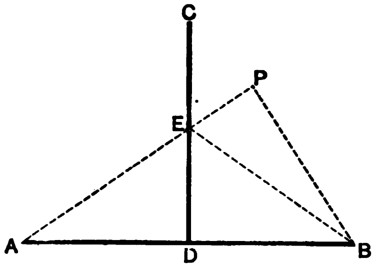
$$AD = DB \text{ (Hypothesis)},$$

$$\angle 1 = \angle 2 \text{ (Right Angles)},$$

$$PD = PD \text{ (Common)}.$$

The △ are equal in all respects by § 30.

∴ $PA = PB$, and *P* is equally distant from *A* and *B*. Q.E.D.



II. **Let** *CD* be ⊥ to *AB* at its middle point *D*, and *P* be any point without *CD*.

To Prove *P* unequally distant from *A* and *B*.

Draw *PA* and *PB*.

[It is required to prove *PA* and *PB* unequal.]

Proof. One of these lines, as *PA*, will intersect the perpendicular *CD* in some point, as *E*.

Draw *EB*.

$$PB < PE + EB. \qquad \text{Axiom 14.}$$

$$EB = EA. \qquad \text{By Case I.}$$

Substitute *EA* for *EB*.

$$PB < PE + EA.$$

$$PB < PA$$

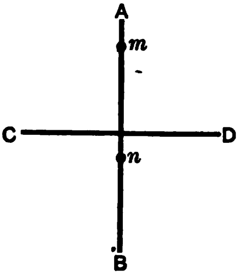
(*PE* and *EA* make up *PA*).

Since *PB* and *PA* are unequal, *P* is unequally distant from *A* and *B*. Q.E.D.

48. Corollary I. *A perpendicular erected to a line at its middle point contains all points that are equally distant from the extremities of the line.*

For, by § 47, any point on the perpendicular is equally distant from the extremities of the line, and any point without the perpendicular is unequally distant from the extremities of the line. Therefore all points that are equally distant from the extremities of the line must be on the perpendicular.

49. Corollary II. *If a line has two of its points each equally distant from the extremities of another line, the first line is perpendicular to the second at its middle point.*



Let *AB* have two of its points *m* and *n* each equally distant from the extremities of *CD*.

To Prove *AB* ⊥ to *CD* at its middle point.

Proof. Suppose a line were drawn ⊥ to *CD* at its middle point.

By § 48 both *m* and *n* must be on this perpendicular.

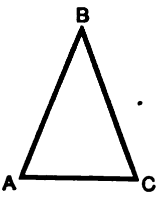
By hypothesis both *m* and *n* are on *AB*.

So the perpendicular and *AB* both pass through *m* and *n*.

By Axiom 13 only one straight line can pass through two given points.

∴ *AB* must coincide with the perpendicular to *CD* at its middle point. Q.E.D.

50. Definitions. In an *isosceles triangle* the angle formed by the two equal sides is called the *vertical angle*. The side opposite this angle is usually called the *base* of the triangle.



51. Exercise. If a perpendicular is erected to the base of an isosceles △ at its middle point, it passes through the vertex of the vertical angle.

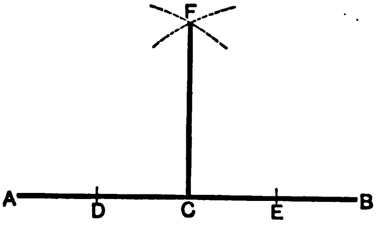
Suggestion. Use § 48.

52. Exercise. If a line is drawn from the vertex of the vertical angle of an isosceles △ to the middle point of the base, it is perpendicular to the base.

Suggestion. Use § 49.

Proposition V. Problem

53. *To erect a perpendicular to a line at a given point on that line.*



Let AB be the given line, and C the given point on the line.

Required to erect a perpendicular to AB at C.

Lay off $CD = CE$.

With D and E as centers, and with a radius greater than DC (one half of DE), describe two arcs intersecting at F.

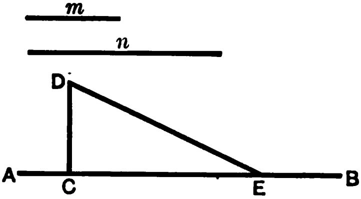
Join F and C.

FC is the required perpendicular. For, F and C are each equally distant from D and E (construction). ∴ by § 49, FC is perpendicular to DE or AB. Q.E.F.

54. Exercise. To construct a R.A. △, having given the two sides about the R.A.

Let m and n be the two given sides.

Required to construct a R.A. △, having m and n as sides about the R.A.



Lay off the indefinite line AB.

At any point of it as C erect $CD \perp AB$, and make CD equal in length to m.

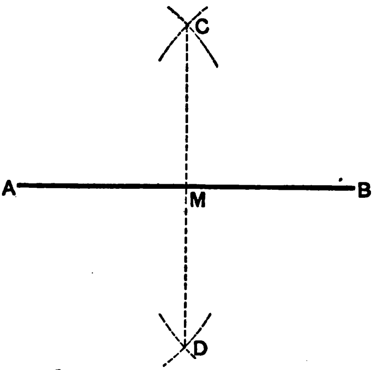
Lay off CE equal to n.

Draw DE.

△ CDE is the required △ because it fulfills all the required conditions; *i.e.* it is right angled at C, and the sides about C are equal respectively to m and n. Q.E.F.

PROPOSITION VI. PROBLEM

55. *To bisect a given line.*



Let AB be the given line.

Required to bisect it.

With A and B as centers, and with any radius greater than one half of AB, describe arcs intersecting at C and D.

Draw CD.

Then will CD bisect AB.

For, the points C and D are each equally distant from the extremities of AB (construction). $\therefore$ CD bisects AB (§ 49).

Q.E.F.

56. EXERCISE. Divide a given line into quarters.

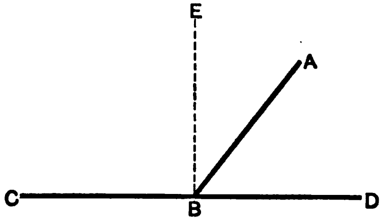
57. EXERCISE. If the radius used for describing the two arcs that intersect at C in the figure of Prop. VI is greater than the radius used for describing the two arcs that intersect at D, will CD bisect AB?

58. EXERCISE. When will the lines AB and CD bisect each other?

59. EXERCISE. In a given line find a point that is equally distant from two given points. When is this problem impossible?

PROPOSITION VII. THEOREM

60. *The sum of the adjacent angles formed by one line meeting another, is two right angles.*



Let AB meet CD at B.

To Prove $\angle ABC + \angle ABD = 2$ R.A.'s.

Proof. Erect BE perpendicular to CD at B. (§ 53.)
By construction $\angle EBC$ and $\angle EBD$ are R. A.'s.

$$\angle ABC = 1 \text{ R.A.} + \angle EBA. \quad (1)$$

$$\angle ABD = 1 \text{ R.A.} - \angle EBA. \quad (2)$$

Adding (1) and (2), $\angle ABC + \angle ABD = 2$ R.A's. Q.E.D.

61. COROLLARY I. *If one of two adjacent angles formed by one line meeting another is a right angle, the other is also a right angle.*

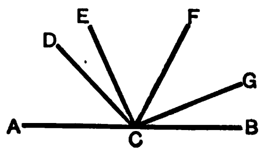
62. COROLLARY II. *If two straight lines intersect each other, and one of the angles formed is a right angle, the other three angles are also right angles.*

63. COROLLARY III. *The sum of all the angles formed at a point in a line, and on the same side of the line, is two right angles.*

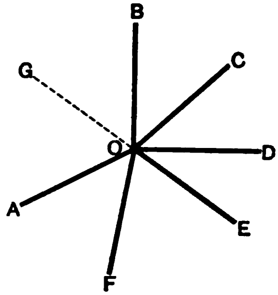
SUGGESTION. Show that the sum of all the angles at C equals

$$\angle FCA + \angle FCB$$

or $$\angle GCA + \angle GCB, \text{ etc.}$$



64. COROLLARY IV. *The sum of all the angles formed about a point is four right angles.*



SUGGESTION. Prolong one of the lines, as OE, to G. Then apply § 63 to the angles on each side of GE.

65. DEFINITION. If two angles are together equal to two right angles, they are called *supplementary angles.* Each angle is the *supplement* of the other.

Adjacent angles formed by one line meeting another are *supplementary adjacent angles.*

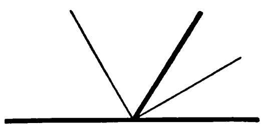
66. DEFINITION. If two angles are together equal to one right angle, they are called *complementary angles.* Each angle is the *complement* of the other.

67. EXERCISE. Find the supplement and also the complement of each of the following angles: $\frac{2}{3}$ R.A., $\frac{1}{6}$ R.A., $\frac{1}{2}$ R.A.

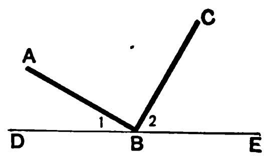
Find the value of each of two supplementary angles, if one is five times the other.

68. EXERCISE. Given an angle, construct its supplement and also its complement.

69. EXERCISE. Prove that the bisectors of two supplementary adjacent angles are perpendicular to each other.



70. EXERCISE. Through the vertex of a right angle a line is drawn outside of the angle. What is the sum of the two acute angles formed? [$\angle 1 + \angle 2 = ?$]



71. EXERCISE. Find the supplement of the complement of $\frac{2}{3}$ R.A., also the complement of the supplement of $1\frac{2}{7}$ R.A.

72. Definition. One proposition is the *converse* of another, when the hypothesis and conclusion of one are respectively the conclusion and hypothesis of the other.

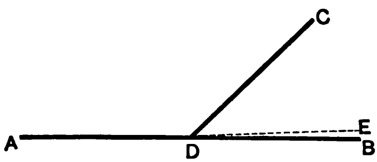
The converse of a proposition is not necessarily true.

We shall prove later (see § 85) that "if the sides of one triangle are equal respectively to the sides of another, the angles of the first triangle are equal respectively to those of the second."

Show, by drawing triangles, that the converse of this proposition, *i.e.* "if the angles of one triangle are equal respectively to the angles of another, the sides of the first triangle are equal respectively to those of the second," is not necessarily true.

Proposition VIII. Theorem (Converse of Prop. VII.)

73. ***If the sum of two adjacent angles is two right angles, their exterior sides form a straight line.***



Let $\angle CDA + \angle CDB = 2$ R.A.'s.

To Prove AD and DB form a straight line.

Proof. Suppose DB is not the prolongation of AD, and that some other line, as DE, is.

By § 60 $\angle CDA + \angle CDE$ would equal 2 R.A.'s.

By hypothesis $\angle CDA + \angle CDB = 2$ R.A.'s.

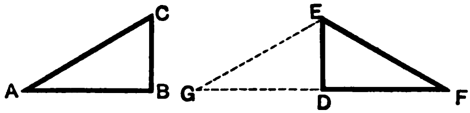
By Axiom 1., $\angle CDA + \angle CDE$ would equal $\angle CDA + \angle CDB$.

Whence $\angle CDE$ would equal $\angle CDB$.

This contradicts Axiom 11.

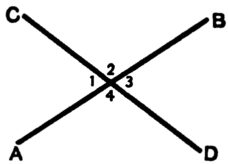
Therefore the supposition that DB is not the prolongation of AD is false, and AD and DB form a straight line. Q.E.D.

74. Exercise. *ABC* and *DEF* are R.A. △ equal in all respects, right angled at *B* and *D*. Place △ *ABC* in the position of △ *GED*.



Prove that *GD* and *DF* form a straight line.

75. Definition. If two lines intersect each other, the opposite angles formed are called *vertical angles*. ∠1 and ∠3 are vertical angles, as are also ∠2 and ∠4.

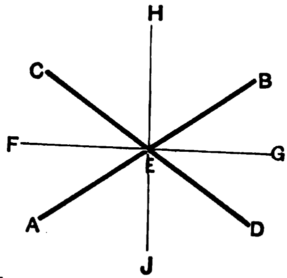


76. Exercise. The bisectors of two opposite angles form a straight line.

Let *FE*, *HE*, *GE*, and *JE* be the bisectors of ∠s *AEC*, *CEB*, *BED*, and *DEA* respectively.

To Prove that *FE* and *EG* form a straight line, and *HE* and *EJ* form a straight line.

Suggestion. Use §§ 69 and 73.



Proposition IX. Theorem

77. *If two straight lines intersect, the opposite or vertical angles are equal.*

Let *AB* and *CD* intersect.

To Prove ∠1 = ∠3
and ∠2 = ∠4.

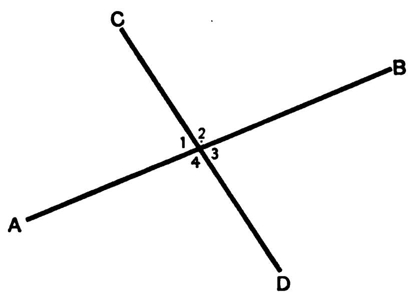
Proof.

∠1 + ∠2 = 2 R.A.'s. (authority ?)

∠2 + ∠3 = 2 R.A.'s. (?)

∠1 + ∠2 = ∠2 + ∠3. (?)

∠1 = ∠3. (?)



In the same manner prove ∠2 = ∠4. Q.E.D.

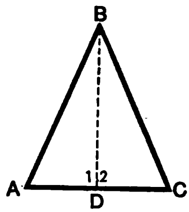
78. Exercise. One angle formed by two intersecting lines is $\frac{2}{3}$ R.A. Find the other three.

79. Exercise. The bisector of an angle bisects its vertical angle.

80. Exercise. Two lines intersect, making the sum of one pair of vertical angles equal to five times the sum of the other pair of vertical angles. Find the values of the four angles.

Proposition X. Theorem

81. *In an isosceles triangle, the angles opposite the equal sides are equal.*



Let ABC be an isosceles $\triangle$, having $AB = BC$.

To Prove $\angle A = \angle C$.

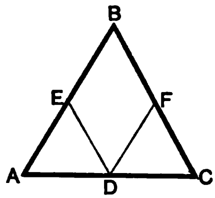
Proof. Draw BD bisecting AC. (§ 55.)

B and D are each equally distant from A and C; $\therefore \angle 1$ and $\angle 2$ are R.A.'s. (?)

Show that $\triangle$s ABD and BDC are equal in all respects.

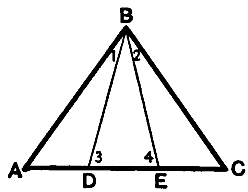
Whence $\angle A = \angle C$. Q.E.D.

82. Corollary. *An equilateral triangle is equiangular.*



83. Exercise. ABC is an isosceles triangle. D is the middle point of the base AC. E and F are the middle points of the equal sides AB and BC.

Prove $DE = DF$.



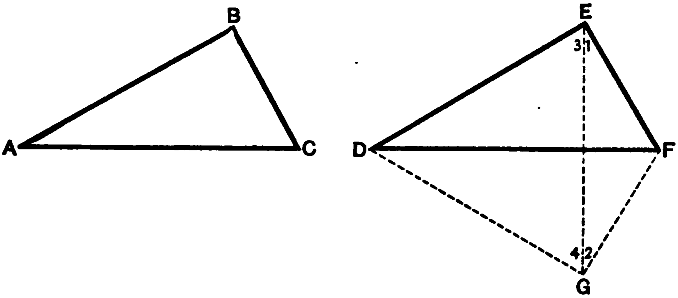
84. Exercise. ABC is an isosceles triangle having $AB = BC$.

BD and BE are drawn making $\angle 1 = \angle 2$.

Prove $\angle 3 = \angle 4$.

Proposition XI. Theorem

85. ***If two triangles have three sides of the one equal respectively to three sides of the other, the triangles are equal in all respects.***



Let ABC and DEF be two $\triangle$, having $AB = DE$, $BC = EF$, and $AC = DF$.

To Prove $\triangle$ ABC and DEF equal in all respects.

Proof. Place $\triangle ABC$ so that AC shall coincide with DF, A falling on D and C on F, and the vertex B falling at G, on the opposite side of the base from the vertex E.

Draw EG.

Prove $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$.

Adding, $\angle 1 + \angle 3 = \angle 2 + \angle 4$, or $\angle DEF = \angle DGF$.

Prove $\triangle$ DEF and DGF equal in all respects.

$\therefore$ $\triangle$ DEF and ABC are equal in all respects. Q.E.D.

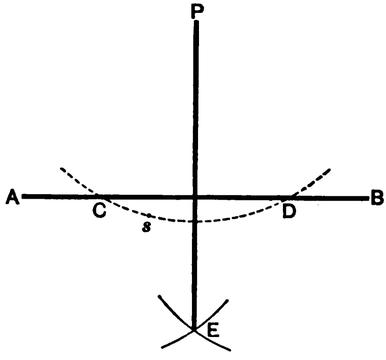
86. Exercise. Construct a triangle having given its three sides.

87. Exercise. Construct a triangle equal to a given triangle.

88. Exercise. Construct a triangle whose sides are in the ratio of 3, 4, and 5.

PROPOSITION XII. PROBLEM

89. *To draw a perpendicular to a line from a point without.*



Let AB be the given line and P the point without.

Required to draw a perpendicular from P to the line AB.

Let s be any point on the opposite side of AB from P.

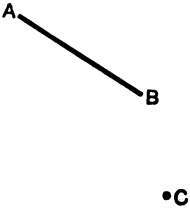
With P as a center, and Ps as a radius, describe an arc intersecting AB at C and D.

With C as a center, and with a radius greater than one half of CD, describe an arc; with D as a center, and with the same radius, describe an arc intersecting the first arc at E.

Draw PE.

Show that PE is perpendicular to CD. Q.E.F.

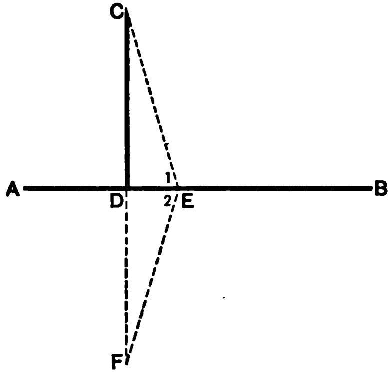
90. EXERCISE. Draw a perpendicular to AB from the point C.



91. EXERCISE. If the line AB (see § 89) were situated at the bottom of this page, and there were no room below it for the point E, how could the perpendicular be drawn?

Proposition XIII. Theorem

92. *From a point without a line only one perpendicular can be drawn to the line.*



Let CD be a $\perp$ from C to AB.

To Prove that CD is the only $\perp$ that can be drawn from C to AB.

Proof. Suppose a second $\perp$, as CE, could be drawn.

Prolong CD until $DF = CD$, and draw EF.

Prove $\triangle$ CDE and FDE equal in all respects.

Whence $\angle 1 = \angle 2$.

But $\angle 1 = 1$ R.A. by supposition.

Show that $\angle 1 + \angle 2 = 2$ R.A.'s.

If the sum of angles 1 and 2 is two R.A.'s, CE and EF form a straight line. (§ 73.)

The points C and F are therefore connected by two straight lines (CDF and CEF), which contradicts (?).

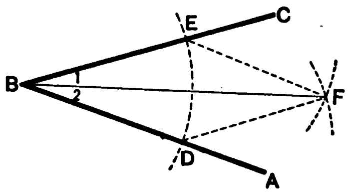
Therefore the supposition that a second $\perp$ could be drawn from C to the line AB is false, and only one $\perp$ can be drawn.

Q.E.D.

93. Exercise. Show that a triangle cannot have two right angles.

PROPOSITION XIV. PROBLEM

94. ***To bisect a given angle.***



Let ABC be any angle.

Required to bisect it.

With B as a center, and with any convenient radius, describe an arc intersecting the sides of the angle at D and E.

With D as a center, and with a radius greater than one half of DE, describe an arc; with E as a center, and with the same radius, describe an arc intersecting this arc at F.

Join B and F.

Then will BF bisect $\angle ABC$.

Draw FE and FD.

Prove $\triangle$ BEF and BDF equal in all respects.

Whence $\angle 1 = \angle 2$, and $\angle ABC$ is bisected. Q.E.F.

95. EXERCISE. At a given point on a line construct an angle equal to $\frac{1}{2}$ R.A.

96. EXERCISE. Divide a given angle into quarters.

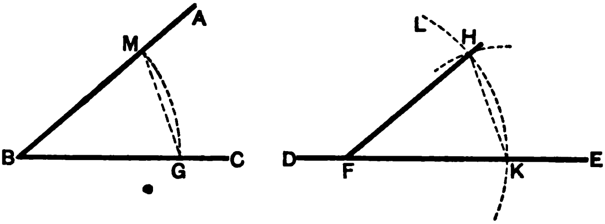
97. EXERCISE. At a given point on a line construct an angle equal to $1\frac{1}{2}$ R.A.'s.

98. EXERCISE. Prove § 81 by drawing BD (see figure of § 81) bisecting angle ABC.

99. EXERCISE. Construct a triangle ABC, making the side AB two inches long, $\angle A = 1$ R.A. and $\angle B = \frac{1}{2}$ R.A.

Proposition XV. Problem

100. *At a point on a line to construct an angle equal to a given angle.*



Let $\angle ABC$ be the given angle, and F the point on the line DE.

Required to construct an angle at F on the line DE that shall equal $\angle ABC$.

With B as a center, and with any radius, describe the arc MG.

With F as a center, and with the same radius, describe the indefinite arc LK, intersecting DE at K.

With K as a center, and with the distance MG as a radius, describe an arc intersecting the arc LK at H.

Draw HF.

Then will $\angle HFK = \angle ABC$.

Draw MG and HK.

Prove ⚠ MBG and HFK equal in all respects.

Whence $\angle B = \angle F$. Q.E.F.

101. Exercise. Construct a triangle having given two sides and the included angle.

102. Exercise. Construct a triangle having given two angles and the included side.

103. Exercise. Construct an angle equal to the sum of two given angles.

104. Exercise. Construct an angle that is double a given angle.

105. Exercise. Construct an angle equal to the difference between two given angles.

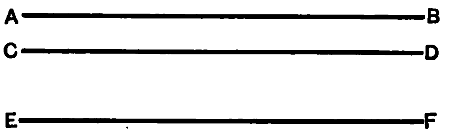
106. Exercise. Draw any triangle. Construct an angle equal to the sum of the angles of this triangle.

From your drawing what do you *infer* the sum of the angles to be? See § 138.

107. Definition. *Parallel lines* are lines lying in the same plane, which do not meet, how far soever they may be prolonged.

Proposition XVI. Theorem

108. *If two lines are parallel to a third line, they are parallel to each other.*



Let AB and CD be ∥ to EF.

To Prove AB and CD ∥ to each other.

Proof. Since AB and CD are in the same plane, if they are not parallel they must meet.

If they do meet we should have two lines drawn through the same point parallel to EF.

This contradicts (?).

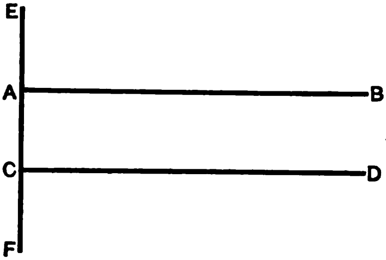
Therefore they cannot meet, and, by definition (§ 107), are parallel. Q.E.D.

109. Exercise. If a line be drawn on this page parallel to the upper edge, show that it is also parallel to the lower edge.

110. Exercise. Give an example of two lines that never meet, how far soever they be prolonged, and yet are not parallel. [*Note.* — To do this the student must leave the province of plane geometry and think of lines in different planes.]

PROPOSITION XVII. THEOREM

111. *If two lines are perpendicular to the same line, they are parallel.*



Let *AB* and *CD* be ⊥ to *EF*.

To Prove *AB* and *CD* ∥ to each other.

Proof. If *AB* and *CD* are not parallel, they will meet at some point. (?)

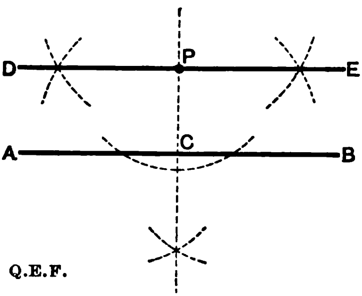
Then we should have two perpendiculars drawn from that point to *EF*.

This contradicts (?).

∴ *AB* and *CD* are parallel. Q.E.D.

112. PROBLEM. Through a given point to draw a line parallel to a given line.

Let *P* be the given point and *AB* the given line.



Required to draw through *P* a parallel to *AB*.

Draw *PC* ⊥ to *AB*.

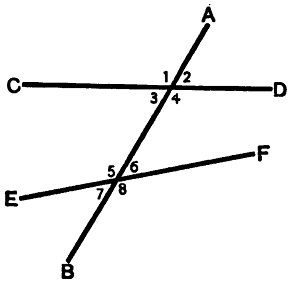
Through *P* draw *DE* ⊥ to *PC*.

Prove *DE* and *AB* parallel. Q.E.F.

113. DEFINITIONS. A straight line that cuts two or more lines is called a *transversal*.

If two lines are cut by a transversal, eight angles are formed, which are named as follows:

The four angles [$\angle 1$, $\angle 2$, $\angle 7$, and $\angle 8$], lying without the two lines, are called *exterior angles.*



The four angles [$\angle 3$, $\angle 4$, $\angle 5$, and $\angle 6$], lying within the two lines, are called *interior angles.*

The two pairs of exterior angles [$\angle 1$ and $\angle 7$, $\angle 2$ and $\angle 8$], lying on the same side of the transversal, are called *exterior angles on the same side.*

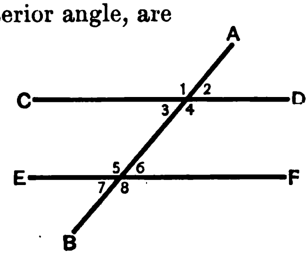
The two pairs of interior angles [$\angle 3$ and $\angle 5$, $\angle 4$ and $\angle 6$], lying on the same side of the transversal, are called *interior angles on the same side.*

The four pairs of angles [$\angle 1$ and $\angle 5$, $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$, $\angle 4$ and $\angle 8$], lying on the same side of the transversal, one an exterior and the other an interior angle, are called *corresponding angles.*

The two pairs of exterior angles [$\angle 1$ and $\angle 8$, $\angle 2$ and $\angle 7$], lying on opposite sides of the transversal, are called *alternate exterior angles.*

The two pairs of interior angles [$\angle 3$ and $\angle 6$, $\angle 4$ and $\angle 5$], lying on opposite sides of the transversal, are called *alternate interior angles.*

The four pairs of angles [$\angle 1$ and $\angle 6$, $\angle 2$ and $\angle 5$, $\angle 3$ and $\angle 8$, $\angle 4$ and $\angle 7$], lying on opposite sides of the transversal, one an exterior and the other an interior angle, are called *alternate exterior and interior angles.*

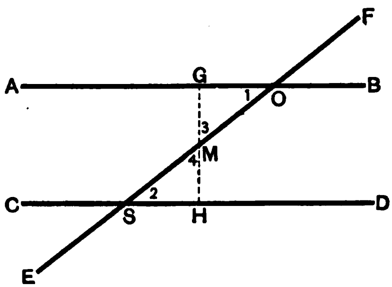


114. Exercise. Show that if any one of the following sixteen equations is true, the other fifteen equations are also true.

1. $\angle 3 = \angle 6$.
2. $\angle 4 = \angle 5$.
3. $\angle 1 = \angle 8$.
4. $\angle 2 = \angle 7$.
5. $\angle 1 = \angle 5$.
6. $\angle 2 = \angle 6$.
7. $\angle 3 = \angle 7$.
8. $\angle 4 = \angle 8$.
9. $\angle 3 + \angle 5 = 2$ R.A.'s.
10. $\angle 4 + \angle 6 = 2$ R.A.'s.
11. $\angle 1 + \angle 7 = 2$ R.A.'s.
12. $\angle 2 + \angle 8 = 2$ R.A.'s.
13. $\angle 1 + \angle 6 = 2$ R.A.'s.
14. $\angle 2 + \angle 5 = 2$ R.A.'s.
15. $\angle 3 + \angle 8 = 2$ R.A.'s.
16. $\angle 4 + \angle 7 = 2$ R.A.'s.

Proposition XVIII. Theorem

115. *If two lines are cut by a transversal, making the alternate interior angles equal, the lines are parallel.*



Let AB and CD be cut by the transversal EF, making $\angle 1 = \angle 2$.

To Prove AB and CD parallel.

Proof. From M, the middle point of SO, draw $MH \perp$ to CD, and prolong MH until it meets AB in some point G.

Prove the $\triangle$s GMO and MSH equal in all respects.

Whence $\angle H = \angle G$.

$\angle H$ is by construction a R.A.

$\therefore \angle G$ is a R.A.

AB and CD are parallel. (?) Q.E.D.

116. Corollary. *If two lines are cut by a transversal, making any one of the following six cases true, the lines are parallel.*

1. The alternate interior angles equal.

2. The alternate exterior angles equal.

3. The corresponding angles equal.

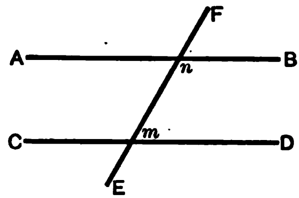
4. The sum of the interior angles on the same side equal to two R.A.'s.

5. The sum of the exterior angles on the same side equal to two R.A.'s.

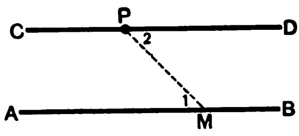
6. The sum of the alternate interior and exterior angles equal to two R.A.'s.

117. Exercise. FE intersects AB and CD, making $\angle m = \frac{2}{3}$ R.A.

What value must $\angle n$ have in order that AB and CD shall be parallel?



118. Exercise. Through a given point to draw a parallel to a given line. (This exercise is to be based on § 115. Another solution was given in § 112.)



[Through the given point P draw any line PM to the given line AB. Through P draw CD, making $\angle 2 = \angle 1$. Prove CD parallel to AB.

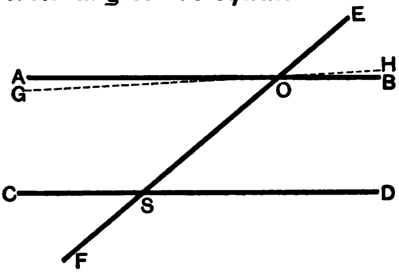
Work this exercise by making the alternate exterior angles equal; also by making the corresponding angles equal.]

119. Exercise. The sum of two angles of a triangle cannot equal two right angles.

120. Exercise. The bisectors of the equal angles 1 and 2 in the figure of § 118, are parallel.

Proposition XIX. Theorem

121. *If two parallels are cut by a transversal, the alternate interior angles are equal.*



Let the parallel lines *AB* and *CD* be cut by the transversal *EF*.

To Prove $\angle AOS = \angle OSD$.

Proof. Suppose $\angle AOS$ is not equal to $\angle OSD$.

Draw *GH* through *O*, making $\angle GOS = \angle OSD$.

GH and *CD* are parallel. (?)

AB and *CD* are parallel. (?)

Through *O* there are two parallels to *CD*, which contradicts (?).

∴ The supposition that $\angle AOS$ and $\angle OSD$ are unequal, etc.

Q.E.D.

122. Corollary I. *If two parallels are cut by a transversal, the six cases of § 116 are true.*

123. Corollary II. *If a line is perpendicular to one of two parallels, it is perpendicular to the other also.*

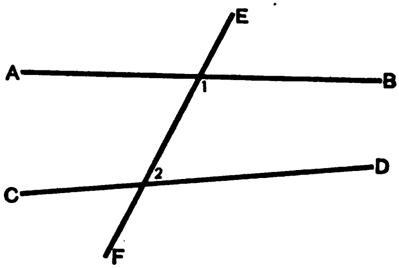
124. Exercise. The bisectors of two alternate exterior angles, formed by a transversal cutting two parallel lines, are parallel.

125. Exercise. If a line joining two parallels is bisected, any other line through the point of bisection, and joining the parallels, is also bisected.

126. Exercise. If *AB* and *CD* are parallel (§ 117), and $\angle n = 1\frac{2}{5}$ R.A., find the values of the other seven angles.

Proposition XX. Theorem

127. ***If two lines are cut by a transversal, making the sum of the interior angles on the same side less than two right angles, the lines will meet if sufficiently produced.***



Let AB and CD be cut by EF, making $\angle 1 + \angle 2 < 2$ R.A.'s.

To Prove that AB and CD will meet.

Proof. If AB and CD do not meet, they are parallel. (?)

If they are parallel, $\angle 1 + \angle 2 = 2$ R.A.'s. (?)

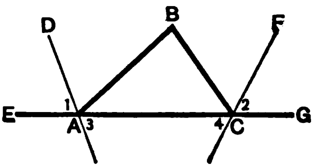
This contradicts (?).

∴ they cannot be parallel and must meet. Q.E.D

128. Corollary. *If two lines are cut by a transversal, making any one of the six cases of § 116 untrue, the lines will meet if sufficiently produced.*

129. Exercise. The bisectors of any two exterior angles of a triangle will meet.

Prove that DA and FC meet.



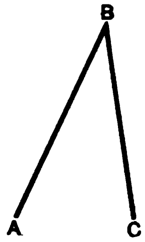
Suggestion. $\angle EAB < 2$ R.A.'s. (?)

$\angle 1 < 1$ R.A. (?)

$\angle 3 < 1$ R.A. (?)

Similarly, $\angle 4 < 1$ R.A.

Whence, $\angle 3 + \angle 4 < 2$ R.A.'s.

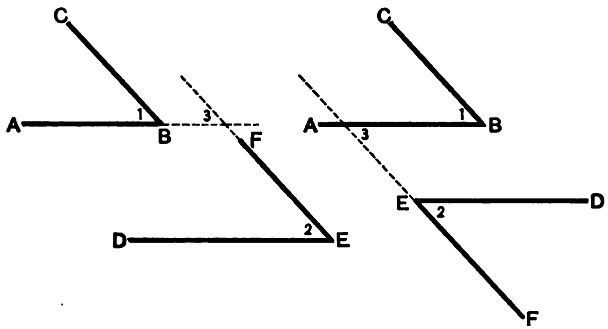


130. Definition. Each angle, viewed from its vertex, has a *right side* and a *left side*.

AB is the right side of $\angle ABC$, and *BC* is its left side.

PROPOSITION XXI. THEOREM

131. *If two angles have their sides parallel, right side to right side, and left side to left side, the angles are equal.*



Let $\angle 1$ and $\angle 2$ have their sides parallel, right side to right side, and left side to left side.

To Prove $\angle 1 = \angle 2$.

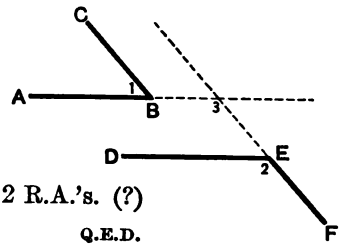
Proof. Prolong *AB* and *EF* until they intersect.

$\angle 1 = \angle 3$. (?)

$\angle 3 = \angle 2$. (?)

$\angle 1 = \angle 2$. (?) Q.E.D

132. Corollary. *If two angles have their sides parallel, right side to left side, and left side to right side, the angles are supplementary.*

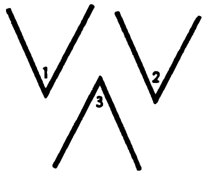


Let ∠1 and ∠2 have their sides parallel, right side to left side and left side to right side.

To Prove ∠1+∠2=2 R.A.'s.

Proof. ∠2 = ∠3, ∠1+∠3=2 R.A.'s. (?)

∴ ∠1 + ∠2 = 2 R.A.'s. Q.E.D.

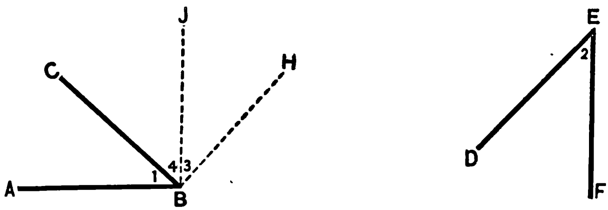


133. Exercise. ∠1 and ∠2 have their sides parallel, right side to right side, etc.

∠2 and ∠3 have their sides parallel, right side to right, etc. Prove that ∠1 = ∠3.

Proposition XXII. Theorem

134. *If the sides of one angle are perpendicular to those of another, right side to right side and left side to left side, the angles are equal.*



Let ∠1 and ∠2 have *DE* ⊥ to *BC* and *FE* ⊥ to *AB*.

To Prove ∠1 = ∠2.

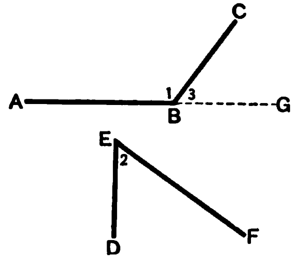
Proof. Draw *BH* ∥ to *ED* and *BJ* ∥ to *FE*. ∠3 = ∠2. (?)

BH is ⊥ to *BC* (?) and *JB* is ⊥ to *AB*. (?)

∠3 + ∠4 = 1 R.A. and ∠1 + ∠4 = 1 R.A.

∠3 = ∠1. (?) ∴ ∠2 = ∠1. (?) Q.E.D.

135. Corollary. *If the sides of one angle are perpendicular to those of another, right side to left side and left side to right side, the angles are supplementary.*



To Prove $\angle 1 + \angle 2 = 2$ R.A.'s.

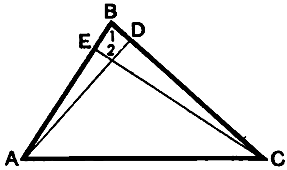
Proof. Prolong AB to G.

Show that $\angle 3 = \angle 2$.

$$\angle 1 + \angle 3 = 2 \text{ R.A.'s.}$$

$$\therefore \angle 1 + \angle 2 = 2 \text{ R.A.'s.}$$

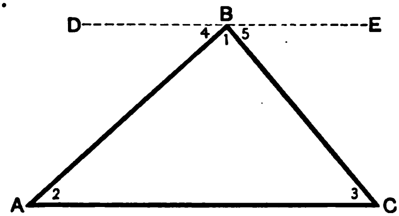
136. Exercise. In $\triangle ABC$, AD is $\perp$ to BC and $CE \perp$ to AB. Compare $\angle 1$ and $\angle 2$.



137. Definition. Two triangles are *mutually equiangular* when the angles of one are equal respectively to the angles of the other.

Proposition XXIII. Theorem

138. *The sum of the interior angles of a triangle is two right angles.*



Let ABC be any $\triangle$.

To Prove $\angle 1 + \angle 2 + \angle 3 = 2$ R.A.'s.

Proof. Draw DE through the vertex B, parallel to AC.

$$\angle 4 = \angle 2 \text{ and } \angle 5 = \angle 3. \quad (?)$$

$$\angle 4 + \angle 1 + \angle 5 = 2 \text{ R.A.'s.} \quad (?)$$

$$\angle 2 + \angle 1 + \angle 3 = 2 \text{ R.A.'s.} \quad (?)$$

Q.E.D.

139. Corollary I. *If two angles of a triangle are known, the third can be found by subtracting their sum from two right angles.*

140. Corollary II. *If two angles of one triangle are equal respectively to two angles of another, the third angles are equal, and the triangles are mutually equiangular.*

141. Corollary III. *A triangle can contain only one right angle; and it can contain only one obtuse angle.*

142. Corollary IV. *In a right-angled triangle, the sum of the acute angles is one right angle.*

143. Corollary V. *Since an equilateral triangle is also equiangular, each angle is two thirds of a right angle.*

144. Corollary VI. *An exterior angle of a triangle (formed by prolonging a side) is equal to the sum of the two opposite interior angles of the triangle.*

145. Exercise. One of the acute angles of a R.A. △ is $\frac{3}{7}$ R.A. What is the other?

146. Exercise. Find the angles of a △, if the second is twice the first, and the third is three times the second.

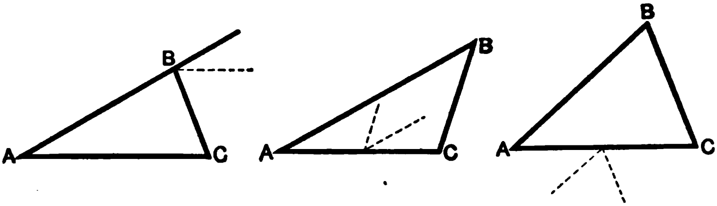
147. Exercise. Find the angles of an isosceles △, if a base angle is one half the vertical angle.

148. Exercise. Given two angles of a triangle, construct the third.

149. Exercise. Prove that the bisectors of the acute angles of an isosceles right-angled triangle make with each other an angle equal to $1\frac{1}{2}$ R.A.'s.

150. Exercise. Prove that the bisector of an exterior vertical angle of an isosceles triangle is parallel to the base.

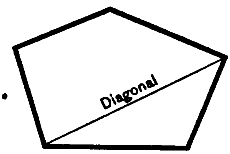
151. Exercise. Prove § 138, using these figures.



152. Definitions. A portion of a plane bounded by straight lines is called a *polygon.*

The bounding line of a polygon is its *perimeter.*

A *diagonal* of a polygon is a straight line joining any two of its vertices that are not consecutive.



A three-sided polygon is a *triangle;* a four-sided polygon is a *quadrilateral;* a five-sided polygon is a *pentagon;* a six-sided polygon is a *hexagon;* an eight-sided polygon is an *octagon;* a ten-sided polygon is a *decagon;* and a fifteen-sided polygon is a *pentedecagon.*

A polygon whose angles are equal is an *equiangular polygon.*

A polygon whose sides are equal is an *equilateral polygon.*

A polygon that is both equilateral and equiangular is a *regular polygon.*

153. Exercise. Show that an equilateral triangle is regular.

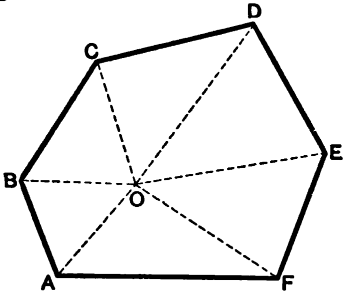
154. Exercise. Show, by drawings, that an equilateral quadrilateral is not necessarily regular.

155. Exercise. How many diagonals can be drawn in a triangle? In a quadrilateral? In a hexagon?

156. Exercise. How many diagonals can be drawn from one vertex in a polygon of n sides? How many from all the vertices?

Proposition XXIV. Theorem

157. *The sum of the interior angles of a polygon is twice as many right angles as the polygon has sides, less four right angles*



Let $ABC \dots F$ be a polygon of n sides.

To Prove that the sum of its interior angles is $(2n-4)$ R.A.'s.

Proof. From any point within the polygon, as O, draw lines to all the vertices.

The polygon is now divided into n ⊿s. (?)

The sum of the angles of each △ is 2 R.A.'s. (?)

The sum of the angles of the n ⊿s is $2n$ R.A.'s. (?)

The sum of the angles of the polygon is equal to the sum of the angles of the ⊿s, diminished by the sum of the angles about O; that is, by 4 R.A.'s.

∴ the sum of the angles of the polygon is $(2n-4)$ R.A.'s.

Q.E.D.

158. Corollary. *The value of each angle of an equiangular polygon of n sides is* $\frac{2n-4}{n}$ *R.A.'s.*

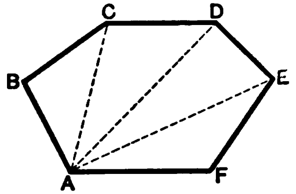
159. Exercise. What is the sum of the interior angles of a quadrilateral? Of a pentagon? Of a hexagon? Of a polygon of 100 sides?

160. Exercise. How many sides has the polygon in which the sum of the interior angles is 20 R.A.'s? 26 R.A.'s? 98 R.A.'s? $(2s-4)$ R.A.'s?

161. Exercise. How many sides has the equiangular polygon in which one angle is $\frac{2}{3}$ R.A.? 1 R.A.? $1\frac{2}{3}$ R.A.? $1\frac{11}{15}$ R.A.?

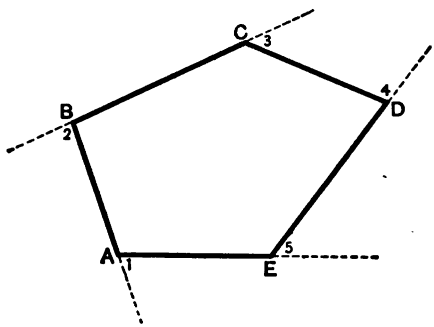
162. Exercise. How many sides has the equiangular polygon in which the sum of four angles is 6 R.A.'s?

163. Exercise. Prove § 157, using this figure. Show that the polygon is divided into $n-2$ triangles, the sum of the angles of which is equal to the sum of the angles of the polygon.



Proposition XXV. Theorem

164. *The sum of the exterior angles of a polygon, formed by prolonging one side at each vertex, is four R.A.'s.*



Let $AB \ldots E$ be a polygon of n sides.

To Prove that the sum of its exterior angles 1, 2, 3, etc., is 4 R.A.'s.

Proof. The sum of each exterior angle and its adjacent interior angle is 2 R.A.'s. (?)

$2n$ R.A.'s is the sum of all exterior and interior angles. (?)

$(2n-4)$ R.A.'s is the sum of the interior angles. (?)

4 R.A.'s is the sum of the exterior angles. (?) Q.E.D.

165. Scholium. It is indifferent which side is prolonged at any vertex, as the exterior angles formed at any vertex by prolonging both sides are equal.

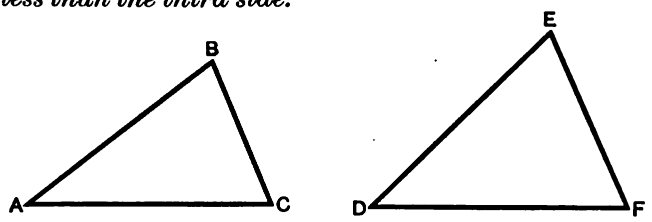
166. Exercise. How many sides has the polygon in which the sum of the interior angles is five times the sum of the exterior angles?

167. Exercise. Complete the following table. The polygons are equiangular.

No. of Sides.	Value of each Interior Angle.	Value of each Exterior Angle.
3	$\frac{2}{3}$ R.A.	$\frac{4}{3}$ R.A.
4	1 R.A.	1 R.A.
5	——	——
⋮	——	——
12	$1\frac{2}{3}$ R.A.	$\frac{1}{3}$ R.A.

Proposition XXVI. Theorem

168. *The sum of two sides of a triangle is greater than the third side, but the difference of two sides of a triangle is less than the third side.*



Let ABC be any $\triangle$.

To Prove $AB + BC > AC$.

Proof. Apply axiom 14. **Q.E.D.**

Let DEF be any $\triangle$.

To Prove $DE - EF < DF$.

Proof. $DE < DF + EF$. (?)

Subtract EF from both members. (?)

$DE - EF < DF$. **Q.E.D**

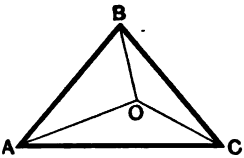
169. Exercise. Can a triangle have for its sides 6 in., 7 in., and 15 in.?

170. Exercise. Two sides of a triangle are 5 ft. and 7 ft. Between what limits must the third side lie?

171. Exercise. Each side of a triangle is less than the semi-perimeter.

172. Exercise. The sum of the lines drawn from a point within a triangle to the three vertices is greater than the semi-perimeter.

Prove $OA + OB + OC > \frac{1}{2}(AB + BC + CA)$.



173. Definition. A *medial line* of a triangle (or simply a *median*) is a line drawn from any vertex of the triangle to the middle point of the opposite side.

174. Exercise. A median to one side of a triangle is less than one half the sum of the other two sides.

To prove $BD < \frac{1}{2}(AB + BC)$.

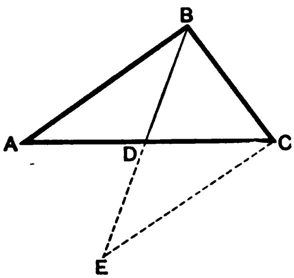
Prolong BD until $DE = BD$.

Draw CE.

Prove $\triangle s\ ABD$ and DCE equal, whence $EC = AB$.

$$BC + CE > BE. \quad (?)$$

Divide both members by 2, recollecting that $BD = DE$ and $EC = AB$.

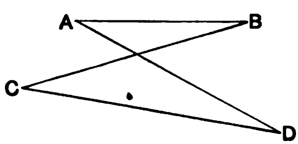


175. Exercise. The sum of the three medians of a triangle is less than its perimeter.

Suggestion. Use the preceding exercise.

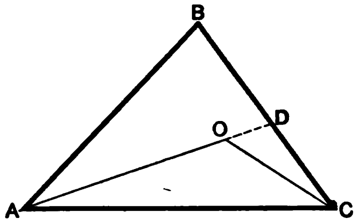
176. Exercise. The lines AB and CD have their extremities joined by CB and AD.

Prove $CB + AD > AB + CD$.



PROPOSITION XXVII. THEOREM

177. ***If from a point within a triangle two lines are drawn to the extremities of a side, their sum is less than that of the two remaining sides of the triangle.***



Let ABC be any $\triangle$, O any point within, and OA and OC lines drawn to the extremities of AC.

To Prove $OA + OC < AB + BC$.

Proof. Prolong AO to D.

$$AB + BD > AO + OD. \quad (?)$$

$$OD + DC > OC. \quad (?)$$

Add these inequalities and show that $AB + BC > AO + OC$.

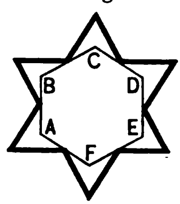
Q.E.D.

178. EXERCISE. Prove $\angle AOC > \angle ABC$.

Suggestion. Show that $\angle AOC > \angle ODC$ and $\angle ODC > \angle ABC$. Give another proof for this exercise without prolonging AO.

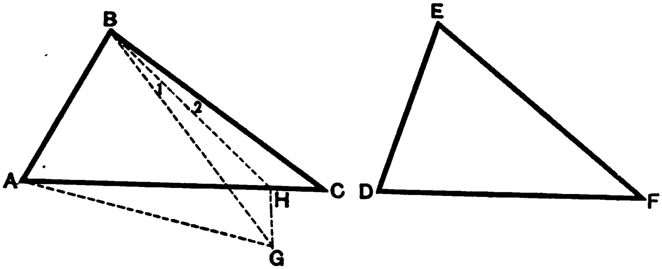
179. EXERCISE. The sum of the lines drawn from a point within a triangle to the three vertices is less than the perimeter of the triangle.

180. EXERCISE. Prove that the perimeter of the star is greater than that of the polygon $ABCDEF$.



PROPOSITION XXVIII. THEOREM

181. *If two triangles have two sides of the one equal respectively to two sides of the other, and the included angles unequal, the third sides are unequal, and the greater third side belongs to the triangle having the greater included angle.*



Let the $\triangle s\ ABC$ and DEF have

$$AB = DE,\ BC = EF$$

and $$\angle B > \angle E.$$

To Prove $$AC > DF.$$

Proof. Of the two sides, AB and BC, let AB be the one which is not the larger.

Draw BG, making $\angle ABG = \angle E$; prolong BG, making $BG = EF$.

Draw AG.

Prove $\triangle ABG = \triangle DEF$, whence $AG = DF$.

Draw BH bisecting $\angle GBC$.

Draw GH.

Prove $\triangle GBH = \triangle HBC$. Whence $HG = HC$.

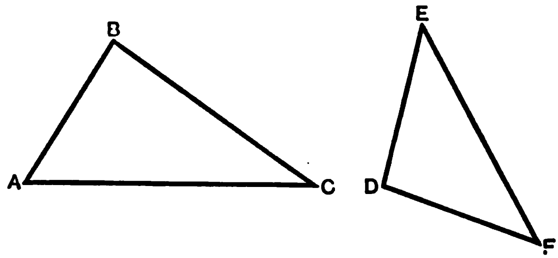
$$AH + HG > AG. \quad (?)$$

$$AC > AG. \quad (?)$$

$$AC > DF. \quad (?)$$

Q.E.D

182. Converse. **If two triangles have two sides of the one equal respectively to two sides of the other, and the third sides unequal, the included angles are unequal, and the greater included angle belongs to the triangle having the greater third side.**



Let $\triangle ABC$ and DEF have

$$AB = DE,\ BC = EF,$$

and $$AC > DF.$$

To Prove $$\angle B > E.$$

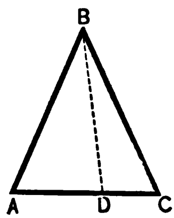
Proof. $\angle B = \angle E$, or $\angle B < \angle E$, or $\angle B > \angle E$.

Show that $\angle B$ cannot equal $\angle E$.
Show that $\angle B$ cannot be less than $\angle E$.

$$\therefore \angle B > \angle E.$$ **Q.E.D.**

183. Exercise. B is fifty miles west of A. C is forty miles north of B, and D is forty miles southeast of B. Show that C is a greater distance from A than D is.

184. Exercise. In the isosceles triangle ABC, BD is drawn to a point D on the base AC so that $AD > DC$.

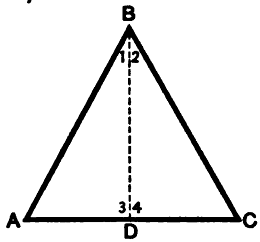


Prove $\angle BDC > \angle ADB$.

Suggestion. Compare $\triangle ABD$ and DBC, using § 182. Then compare $\triangle ADB$ and DBC, using § 144.

Proposition XXIX. Theorem

185. ***If two angles of a triangle are equal, the sides opposite them are equal.***



Let ABC be a $\triangle$ having $\angle A = \angle C$.

To Prove $AB = BC$.

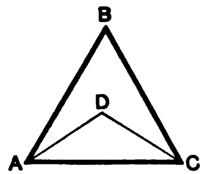
Proof. Draw BD bisecting $\angle B$.

Prove $\triangle$s ABD and BDC mutually equiangular.

Prove $\triangle$s ABD and BDC equal in all respects.

Whence $AB = BC$. Q.E.D.

186. Corollary. *An equiangular triangle is equilateral.*

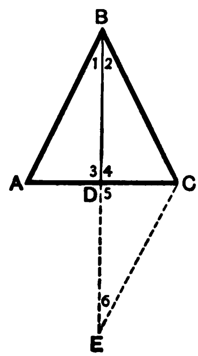


187. Exercise. ABC is an isosceles triangle having $AB = BC$.

AD and DC bisect $\angle A$ and $\angle C$ respectively.

Prove $AD = DC$.

188. Exercise. If the bisector of an angle of a triangle bisects the opposite side, it is also perpendicular to that side, and the triangle is isosceles.



Let BD bisect $\angle B$ and also bisect AC.

To Prove $BD \perp$ to AC, and $\triangle ABC$ isosceles.

Suggestion. Prolong BD until $DE = BD$.

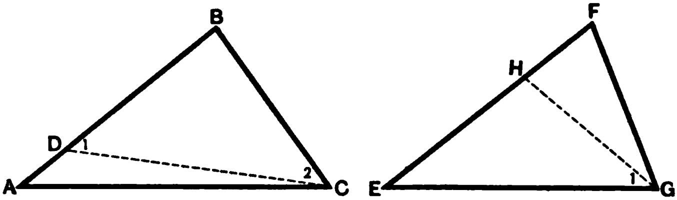
Prove $\triangle ABD = \triangle DEC$.

Whence $\angle 1 = \angle 6$.

Prove $\triangle BCE$ isosceles.

PROPOSITION XXX. THEOREM

189. *If two sides of a triangle are unequal, the angles opposite to them are unequal, the greater angle being opposite the greater side; and conversely, if two angles of a triangle are unequal, the sides opposite them are unequal, the greater side lying opposite the greater angle.*



Let ABC be a $\triangle$ having $AB > BC$.

To Prove $\angle C > \angle A$.

Proof. On AB lay off $BD = BC$ and draw DC.

$\angle 1 = \angle 2$. (?)

$\angle 1 > \angle A$. (?)

$\angle 2 > \angle A$. (?)

$\angle C > \angle A$. (?) Q.E.D.

Let EFG be a $\triangle$ having $\angle G > \angle E$.

To Prove $EF > FG$.

Proof. Draw GH making $\angle 1 = \angle E$.

$HG + HF > FG$. (?)

$HG = EH$. (?)

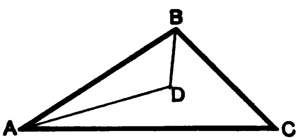
$EF > FG$. (?) Q.E.D.

190. EXERCISE. Prove the converse to this proposition indirectly. Show that EF can neither be equal to FG nor less than FG, and must consequently be greater than FG.

191. EXERCISE. ABC is a triangle having $AC > BC$.

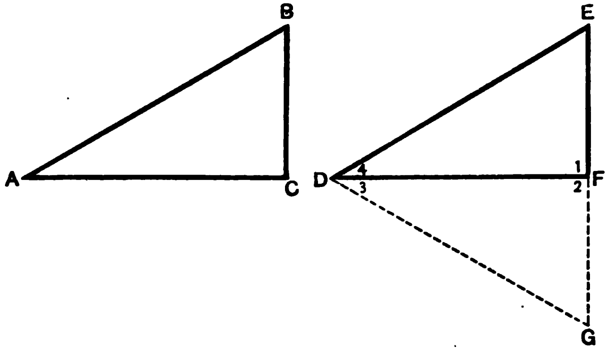
AD bisects $\angle A$ and BD bisects $\angle B$.

Prove $AD > BD$.



Proposition XXXI. Theorem

192. *If two right-angled triangles have the hypotenuse and a side of one equal respectively to the hypotenuse and a side of the other, the triangles are equal in all respects.*



Let ABC and DEF be two R.A.⊿ having hypotenuse $AB =$ hypotenuse DE, and $AC = DF$.

To Prove the ⊿ ABC and DEF equal in all respects.

Proof. Place $\triangle ABC$ so that AC coincides with its equal DF, A falling at D, and C at F, and the vertex B falling at some point G on the opposite side of the base DF from E.

Show that EF and FG form a straight line.

Show (in the $\triangle GDE$) that $\angle G = \angle E$.

$$\angle 3 = \angle 4. \quad (?)$$

⊿ DFG and DFE are equal in all respects. (?)

⊿ ABC and DFE are equal in all respects. Q.E.D.

193. Exercise. If a line is drawn from the vertex of an isosceles triangle ⊥ to the base, it bisects the base and the vertical angle.

194. Definitions. A quadrilateral having its opposite sides parallel is called a *parallelogram.*

A quadrilateral with one pair of parallel sides is a *trapezoid.*

A quadrilateral with no two of its sides parallel is a *trapezium.*

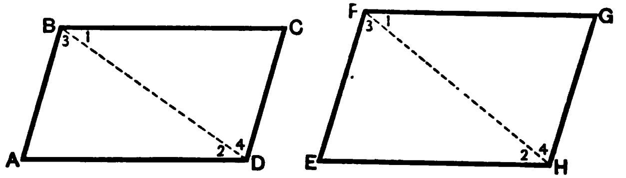
A parallelogram whose angles are right angles is a *rectangle.*

A parallelogram whose angles are oblique angles is a *rhomboid.*

A *square* is an equilateral rectangle; and a *rhombus* is an equilateral rhomboid.

Proposition XXXII. Theorem

195. *The opposite sides of a parallelogram are equal; and conversely, if the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Let $ABCD$ be a parallelogram.

To prove $AB = CD$ and $BC = AD$.

Proof. Draw the diagonal BD.

$\angle 1 = \angle 2$. (?)

$\angle 3 = \angle 4$. (?)

Show that $\triangle ABD = \triangle BCD$.

Whence $AB = CD$ and $BC = AD$. Q.E.D.

Let $EFGH$ be a quadrilateral having $EF = GH$ and $FG = EH$.

To prove $EFGH$ a parallelogram.

Proof. Draw the diagonal FH.

Prove $\triangle EFH = \triangle FGH$.

Whence $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$.

Since $\angle 1 = \angle 2$, FG and EH are parallel. (?)
Similarly EF is parallel to GH.
$EFGH$ is a parallelogram. Q.E.D.

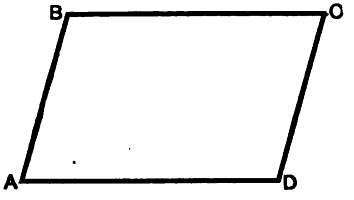
196. Corollary I. *A diagonal of a parallelogram divides it into two triangles equal in all respects.*

197. Corollary II. *Two parallelograms are equal if they have two adjacent sides and the included angle of one equal respectively to two adjacent sides and the included angle of the other.*

198. Corollary III. *Parallels included between two parallels and limited by them, are equal.*

Proposition XXXIII. Theorem

199. *The opposite angles of a parallelogram are equal; and conversely, if the opposite angles of a quadrilateral are equal, the figure is a parallelogram.*



Let $ABCD$ be a parallelogram.

To Prove $\angle A = \angle C$ and $\angle B = \angle D$.

Proof. Show by § 131 that $\angle A = \angle C$ and $\angle B = \angle D$.

Q.E.D.

Conversely. In the quadrilateral $ABCD$ let $\angle A = \angle C$ and $\angle B = \angle D$.

To Prove $ABCD$ a parallelogram.

Proof. $\angle A + \angle B + \angle C + \angle D = 4$ R.A.'s. (?)

$\angle A = \angle C$ and $\angle B = \angle D$.

$2\angle A + 2\angle B = 4$ R.A.'s. (?)

$\angle A + \angle B = 2$ R.A.'s. (?)

BC and AD are parallel. (?)

Similarly prove AB and CD parallel.

$ABCD$ is a parallelogram. (?) Q.E.D.

200. Corollary. *The consecutive angles of a parallelogram are supplementary; and conversely, if the consecutive angles of a quadrilateral are supplementary, the figure is a parallelogram.*

201. Exercise. If one of the angles of a parallelogram is a right angle, the other three are also right angles.

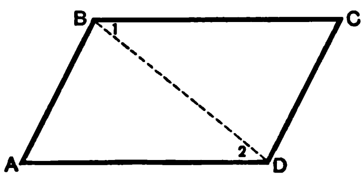
202. Exercise. If one angle of a parallelogram is $\frac{5}{8}$ R.A., how large are the others?

203. Exercise. If two sides of a quadrilateral are parallel, and a pair of opposite angles are equal, the figure is a parallelogram.

204. Exercise. If an angle in one parallelogram is equal to an angle in another, the remaining angles are equal each to each.

Proposition XXXIV. Theorem

205. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Let $ABCD$ be a quadrilateral having BC and AD equal and parallel.

To Prove $ABCD$ a parallelogram.

Proof. Draw the diagonal BD.

$$\triangle ABD = \triangle BCD. \quad (?)$$

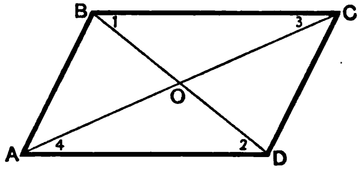
Whence $$AB = CD.$$

Prove $ABCD$ a parallelogram. [§ 195. Converse.] Q.E.D.

206. Exercise. The line joining the middle points of two opposite sides of a parallelogram is parallel to each of the other two sides and equal to either of them.

Proposition XXXV. Theorem

207. *The diagonals of a parallelogram bisect each other; and conversely, if the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*



Let $ABCD$ be a parallelogram, DB and AC its diagonals.

To Prove $BO = OD$ and $AO = OC$.

Proof. Prove $\triangle BOC = \triangle AOD$, whence $BO = OD$ and $AO = OC$.

Q.E.D.

Conversely. In the quadrilateral $ABCD$,

Let $AO = OC$ and $BO = OD$.

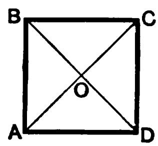
To Prove $ABCD$ a parallelogram.

Proof. Prove $\triangle BOC = \triangle AOD$,

whence $\angle 1 = \angle 2$ and $BC = AD$.

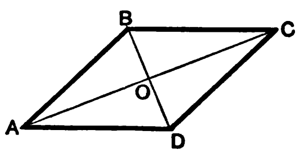
Prove $ABCD$ a parallelogram. (§ 205.) Q.E.D.

208. Corollary I. *The diagonals of a square*



1. *Are equal.*
2. *Bisect each other.*
3. *Are perpendicular to each other.*
4. *Bisect the angles of the square.*

209. Corollary II. *The diagonals of a rhombus*



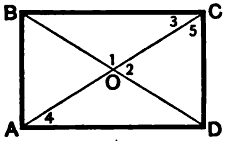
1. *Are unequal.*
2. *Bisect each other.*
3. *Are perpendicular to each other.*
4. *Bisect the angles of the rhombus.*

To prove the diagonals unequal,

first show that $\angle A$ and $\angle D$ of the rhombus are unequal. (They are *supplementary* and *oblique*.)

Then apply § 181 to $\triangle$ ABD and ACD.

210. Corollary III. *The diagonals of a rectangle that is not a square*

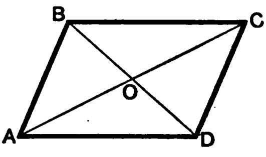


1. *Are equal.*
2. *Bisect each other.*
3. *Are not perpendicular to each other.*
4. *Do not bisect the angles of the rectangle.*

To prove that the diagonals are not perpendicular to each other, apply § 182 to $\triangle$ BOC and COD. (BC and CD are *unequal* because the rectangle is not a square.)

To prove that the diagonals do not bisect the angles of the rectangle, show that $\angle$s 4 and 5 of $\triangle ACD$ are unequal, but $\angle 3 = \angle 4$. (?) $\therefore$ $\angle 3$ and $\angle 5$ are unequal.

211. Corollary IV. *The diagonals of a rhomboid that is not a rhombus*



1. *Are unequal.*
2. *Bisect each other.*
3. *Are not perpendicular to each other.*
4. *Do not bisect the angles of the rhomboid.*

212. Exercise. Any line drawn through the point of intersection of the diagonals of a parallelogram and limited by the sides is bisected at the point.

213. Exercise. If the diagonals of a parallelogram are equal, the figure is a rectangle.

214. Exercise. Given a diagonal, construct a square.

215. Exercise. Given the diagonals of a rhombus, construct the rhombus.

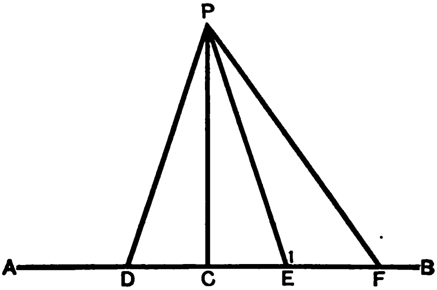
PROPOSITION XXXVI. THEOREM

216. *If from a point without a line a perpendicular is drawn to the line, and oblique lines are drawn to different points of it,*

I. *The perpendicular is shorter than any oblique line.*

II. *Two oblique lines that meet the given line at points equally distant from the foot of the perpendicular are equal.*

III. *Of two oblique lines that meet the given line at points unequally distant from the foot of the perpendicular, the one at the greater distance is the longer.*



I. **Let** *AB* be the given line and *P* the point without, *PC* the ⊥, and *PD* any oblique line.

To Prove $PC < PD$.

Suggestion. Apply § 189, converse, to △ *PCD*.

II. **Let** *PD* and *PE* be oblique lines meeting *AB* at points equally distant from *C*.

To Prove $PD = PE$.

III. **Let** *PF* and *PD* be oblique lines, *F* being at a greater distance from *C* than is the point *D*.

To Prove $PF > PD$.

Suggestion. Show that ∠1 is obtuse. Then apply § 189, converse, to △ *PEF*, recollecting that $PE = PD$.

217. Corollary I. *The perpendicular is the shortest distance from a point to a line, and conversely.*

218. Corollary II. *From a point without a line only two equal lines can be drawn to the line.*

Note. The number of *pairs* of equal lines that can be drawn from a point to a line is of course infinite.

219. Corollary III. *If from a point without a line a perpendicular and two equal oblique lines be drawn, the oblique lines meet the given line at points equally distant from the foot of the perpendicular.*

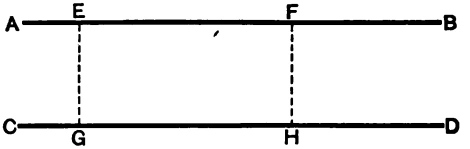
Suggestion. Use § 192.

220. Definition. An *altitude* of a triangle is a perpendicular drawn from the vertex of any angle to the opposite side.

221. Exercise. The sum of the altitudes of a triangle is less than the perimeter.

Proposition XXXVII. Theorem

222. *Two parallels are everywhere equally distant.*



Let AB and CD be two ∥'s.

To Prove that they are everywhere equally distant.

Proof. From any two points on AB, as E and F, draw EG and FH ⊥ to CD.

They are also ⊥ to AB (?), and they measure the distance between the parallels at E and F.

EG and FH are parallel. (?)

EG and FH are equal. (?)

Therefore the parallels are equally distant at E and F.

Since E and F are *any* points on AB, the parallels are *everywhere* equally distant. Q.E.D.

223. Scholium. The term *distance* in geometry means *shortest distance.*

The distance from one point to another is measured on the straight line joining them. (Axiom 14.)

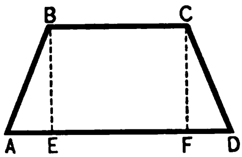
The distance from a point to a line is the perpendicular drawn from that point to the line. (§ 216.)

The distance between two parallels is measured on a line perpendicular to both. (§ 222.)

The distance between two lines in the same plane that are not parallel is zero; for *distance* means *shortest distance,* and the lines will meet if sufficiently produced.

224. Corollary. *If two points are on the same side of a given line and equally distant from it, the line joining the points is parallel to the given line.*

225. Exercise. If the two angles at the extremities of one base of a trapezoid are equal, the two non-parallel sides are equal.



Suggestion. Draw BE and $CF \perp$ to AD. $BE = CF$ (?). Prove ⊿ ABE and CDF equal. Whence $AB = CD$.

226. Exercise. If the two non-parallel sides of a trapezoid are equal, the angles at the extremities of either base are equal.

Suggestion. In the figure of the preceding exercise, prove ⊿ ABE and CFD equal. Whence $\angle A = \angle D$.

227. Exercise. If a quadrilateral has one pair of opposite sides equal and not parallel, and the angles made by these sides with the base equal, the quadrilateral is a trapezoid.

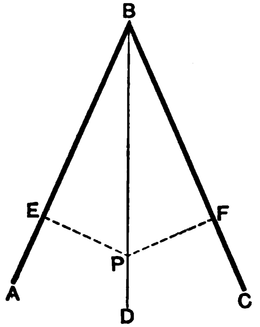
Suggestion. In the figure of § 225, let $AB = CD$ and $\angle A = \angle D$. Prove ⊿ ABE and CFD equal, and then use § 224.

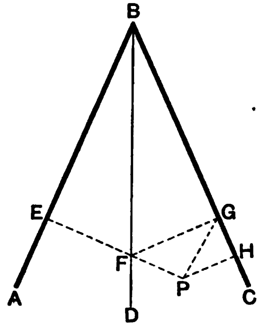
228. Exercise. If two points are on opposite sides of a line, and are equally distant from the line, the line joining them is bisected by the given line.

229. Exercise. If a rectangle and a rhomboid have equal bases and equal altitudes, the perimeter of the rectangle is less than that of the rhomboid.

PROPOSITION XXXVIII. THEOREM

230. *Any point on the bisector of an angle is equally distant from the sides of the angle; and any point not on the bisector is unequally distant from the sides.*





Let ABC be any angle, BD its bisector, and P any point on BD.

To Prove P equally distant from AB and BC.

Proof. Draw PE and PF perpendicular to AB and BC respectively.

Prove $\triangle EPB = \triangle PBF$.

Whence $PE = PF$. **Q.E.D.**

Let ABC be any angle, BD its bisector, and P any point without BD.

To Prove P unequally distant from AB and BC.

Proof. Draw PE and $PH \perp$ to AB and BC respectively.

From F (where PE intersects BD) draw $FG \perp$ to BC.

Draw FP.

$$FP + FG > PG. \quad (?)$$

$$PG > PH. \quad (?)$$

$$FP + FG > PH. \quad (?)$$

$$FE = FG. \quad (?)$$

$$FP + FE > PH. \quad (?)$$

$$PE > PH. \quad (?)$$

Q.E.D.

231. Corollary. *Any point that is equally distant from the sides of an angle is on the bisector.*

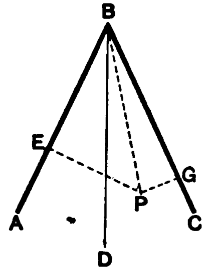
232. Exercise. Prove the second part of § 230 *indirectly.*

Suppose $PE = PG$. Draw PB.

Prove $\triangle PEB = \triangle PBG$.

Whence $\angle PBE = \angle PBG$.

$\therefore$ PB must bisect $\angle ABC$.



233. Definition. The *locus* of a point satisfying a certain condition is the line, lines, or part of a line to which it is thereby restricted; provided, however, that the condition is satisfied by every point of such line or lines, and by no other point.

The bisector of an angle is the locus of points that are equally distant from its sides; for by § 230, all the points on the bisector are equally distant from the sides, and all points without the bisector are unequally distant from the sides.

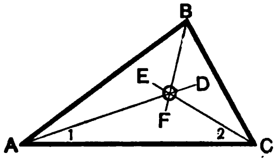
234. Exercise. What is the locus of points that are equally distant from a given point? From two given points?

235. Exercise. What is the locus of points that are equally distant from a given line?

236. Exercise. What is the locus of points that are equally distant from a given circumference?

237. Exercise. The bisectors of the interior angles of a triangle meet in a common point.

To Prove that the bisectors AD, BF, and EC meet in a common point.

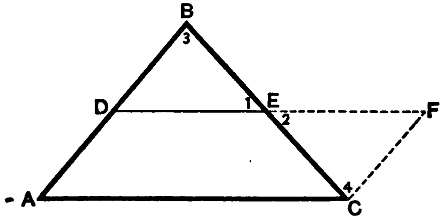


Prove that AD and EC meet. (§ 127.)
Call their point of meeting O.
O is equally distant from AB and AC. (?)
O is equally distant from AC and BC. (?)
$\therefore$ O is equally distant from AB and BC.
O is on the bisector BF. (§ 231.)

Q.E.D

Proposition XXXIX. Theorem

238. *The line joining the middle points of two sides of a triangle is parallel to the third side, and equal to one half of it.*



Let DE join the middle points of AB and BC.

To Prove $DE \parallel$ to AC, and $DE = \frac{1}{2} AC$.

Proof. Prolong DE until $EF = DE$. Draw FC.

Prove ▲ BDE and EFC equal in all respects.

Whence $DB = FC$ and $\angle 3 = \angle 4$.

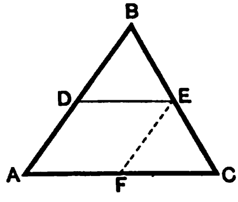
$FC = AD$. (?) FC is $\parallel$ to AD. (?)

$ADFC$ is a parallelogram. (?)

$\therefore$ DE is $\parallel$ to AC.

Prove that $DE = \frac{1}{2} AC$. Q.E.D.

239. Corollary I. *If a line is drawn through the middle point of one side of a triangle parallel to the base, it bisects the other side, and is equal to one half the base.*



Let DE be drawn from the middle point of $BC \parallel$ to AC.

To Prove DE bisects AB, and $DE = \frac{1}{2} AC$.

Proof. Draw $EF \parallel$ to AB.

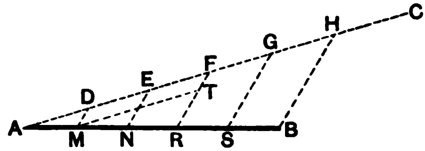
Prove $\triangle DBE = \triangle FEC$.

Whence $EF = DB$ and $DE = FC$

$EF = AD$. (?)

D is the middle point of AB. (?) $DE = \frac{1}{2} AC$. (?) Q.E.D.

240. COROLLARY II. *To divide a line into any number of equal parts.*



Let AB be the given line.

Required to divide it into any number, say five, equal parts.

Draw AC, making any convenient angle with AB.

On AC lay off five equal distances, AD, DE, EF, FG, and GH.

Draw HB.

Draw GS, FR, EN, and DM parallel to HB.

AB is divided into five equal parts.

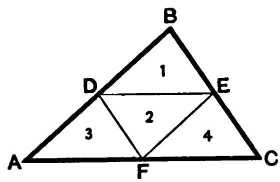
Prove $AM = MN$ (§ 239).

Draw $MT \parallel$ to AC.

Prove $MN = NR$ (§ 239).

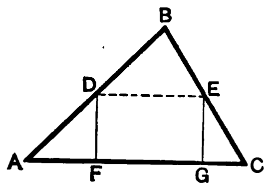
In a similar manner prove $NR = RS$, and $RS = SB$. Q.E.F.

241. EXERCISE. The lines joining the middle points of the three sides of a triangle, divide it into four triangles equal in all respects.



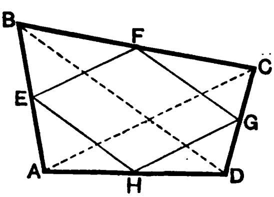
Prove $\triangle 1 = \triangle 2 = \triangle 3 = \triangle 4$.

242. EXERCISE. Perpendiculars drawn from the middle points of two sides of a triangle to the third side are equal.



Prove $DF = EG$.

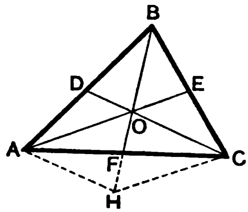
243. EXERCISE. The lines joining the middle points of the sides of a quadrilateral form a parallelogram, equal in area to one half the quadrilateral.



Use § 238 to prove $EFGH$ a parallelogram.

Use § 241 to prove $EFGH = \frac{1}{2} ABCD$.

244. Exercise. The medial lines of a triangle intersect in a common point.



Draw two medial lines AE and CD.

Prove that they meet (§ 127) in some point O.

Draw BO and prolong it.

It is required to show that F is the middle point of AC.

Draw $AH \parallel$ to DC, and prolong BF until it meets AH.

Draw HC.

Prove $BO = OH$, by using $\triangle ABH$.

In $\triangle HBC$, prove OE parallel to HC.

$AOCH$ is a parallelogram. $\therefore$ F is the middle point of AC. Q.E.D.

245. Exercise. The point of intersection of the medial lines divides each median into two segments that are to each other as two is to one.

246. Exercise. Given the middle points of the sides of a triangle, to construct the triangle.

As the variety of exercises in Geometry is practically unlimited, it is impossible to give for their solution any general rules, as may usually be done for problems in Elementary Algebra or Arithmetic. Yet the following hints may be of use to the beginner:

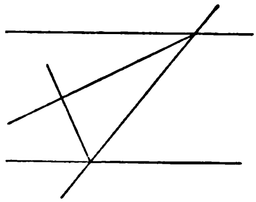
1. Thoroughly digest all the facts of the statement, separating clearly the hypothesis from the conclusion.
2. Draw a diagram expressing all of these facts, including what is to be proved.
3. Draw any auxiliary lines that may seem to be necessary in the proof.[1]
4. Assuming the conclusion to be true, try to deduce from it simpler relations existing between the parts of the figure, and finally some relation that can be established. (This is the *Analysis of the Proposition.*)

[1] The student should remember in drawing auxiliary lines that a straight line may be drawn fulfilling only *two conditions.* Two conditions are said to *determine* a straight line.

5. Then, starting with the relation established, reverse the analysis, tracing it back, step by step, until the conclusion is reached.

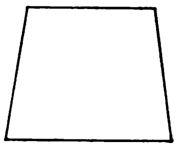
EXERCISES

1. If two angles of a quadrilateral are supplementary, the other two are also supplementary.

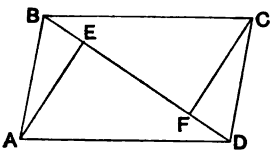


2. Two parallels are cut by a transversal. Prove that the bisectors of two interior angles on the same side are perpendicular to each other.

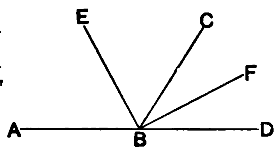
3. An exterior base angle of an isosceles triangle is $1\frac{1}{6}$ R.A.'s. Find the angles of the triangle.



4. If the angles adjacent to one base of a trapezoid are equal, the angles adjacent to the other base are also equal. [§ 122.]

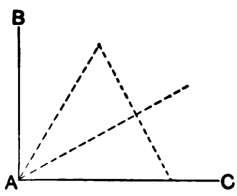


5. In the parallelogram *ABCD*, *AE* and *CF* are drawn perpendicular to the diagonal *BD*. Prove $AE = CF$.



6. *ABC* and *CBD* are two supplementary adjacent angles. *EB* bisects $\angle ABC$, and *BF* is perpendicular to *EB*. Prove that *BF* bisects $\angle CBD$.

7. Construct a right-angled triangle, having given the hypotenuse and one of the acute angles.

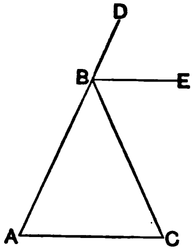


8. Trisect a right angle.

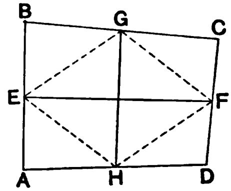
9. Construct an isosceles triangle, having given the base and the vertical angle.

Suggestion. Find the base angles.

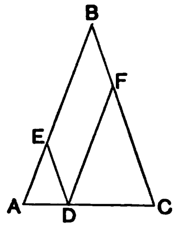
10. *ABC* is an isosceles triangle, and *BE* is parallel to *AC*. Prove that *BE* bisects the exterior angle *CBD*.



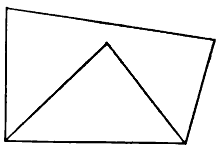
11. The lines joining the middle points of the opposite sides of a quadrilateral bisect each other. [§ 243.]



12. From any point *D* on the base of the isosceles triangle *ABC*, *DE* and *DF* are drawn parallel to the equal sides *BC* and *AB* respectively. Prove that the perimeter of *DEBF* is constant and equals *AB* + *BC*.

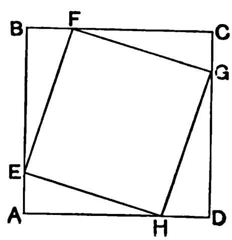


13. The angle formed by the bisectors of two consecutive angles of a quadrilateral is equal to one half the sum of the other two angles. [§§ 138 and 157.]



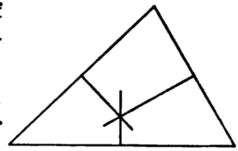
14. How many sides has the polygon the sum of whose interior angles exceeds the sum of its exterior angles by 12 right angles?

15. On the sides of the square *ABCD*, the equal distances *AE*, *BF*, *CG*, and *DH* are laid off. Prove that the quadrilateral *EFGH* is also a square.



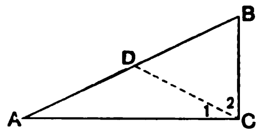
16. The perpendiculars erected to the sides of a triangle at their middle points meet in a common point.

Suggestion. Show that two of the ⊥'s meet. Then show that the third ⊥ passes through their point of meeting. [§ 48.]



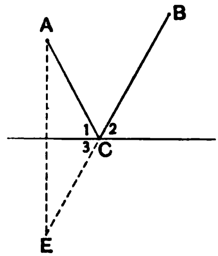
17. The middle point of the hypotenuse of a right-angled triangle is equally distant from the three vertices.

Suggestion. Draw CD, making $\angle 1 = \angle A$. Prove $\angle 2 = \angle B$, and $AD = DC = DB$.



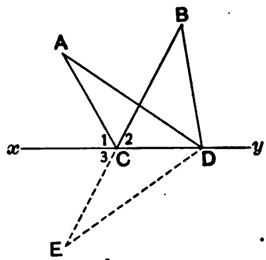
18. The lines joining the middle points of the consecutive sides of a rhombus form a rectangle, which is not a square.

19. From two points on the same side of a line draw two lines meeting in the line and making equal angles with it.



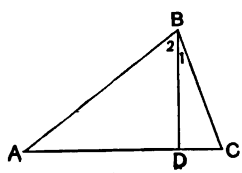
20. Prove that the sum of AC and BC (the lines that make equal angles with xy) is less than the sum of any other pair of lines drawn from A and B and meeting in xy.

Prolong BC until $CE = AC$. Prove $AD = DE$. Then apply § 108 to $\triangle BDE$.

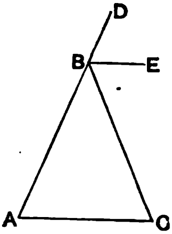


21. If the base of an isosceles triangle is prolonged, twice the exterior angle = 2 R.A.'s + the vertical angle of the triangle.

22. In the triangle ABC, BD is drawn perpendicular to AC. Prove that the difference between $\angle 2$ and $\angle 1$ equals the difference between $\angle A$ and $\angle C$.

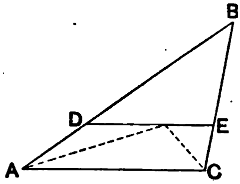


23. Given the sum of the diagonal and a side of a square, construct the square.



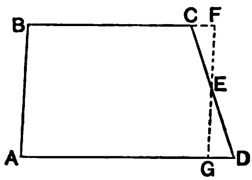
24. If BE is parallel to the base AC of the triangle ABC, and also bisects the exterior angle CBD, prove that the triangle ABC is isosceles.

25. Given the difference between the diagonal and a side of a square, construct the square.



26. Draw DE parallel to the base of the triangle ABC so that $DE = DA + EC$.

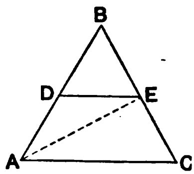
Two constructions. DE may cut the prolonged sides.



27. $ABCD$ is a trapezoid. Through E, the middle of CD, draw FG parallel to BA and meeting BC produced at F.

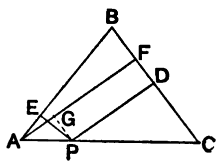
Prove the parallelogram $ABFG$ equal in area to the trapezoid $ABCD$.

28. The angle formed by the bisectors of two angles of an equilateral triangle is double the third angle.



29. In the isosceles triangle ABC draw DE parallel to the base AC, so that $DA = DE = EC$.

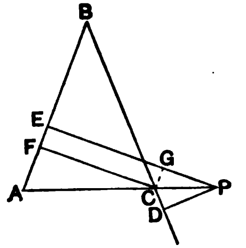
30. If the diagonals of a parallelogram are equal and perpendicular to each other, the figure is a square.



31. If from a point on the base of an isosceles triangle perpendiculars are drawn to the two equal sides, their sum is equal to a perpendicular drawn from either extremity of the base to the opposite side.

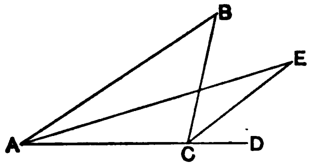
Suggestion. Draw $PG \parallel$ to BC. Prove $\triangle$ AEP and AGP equal.

32. If from a point on the prolonged base of an isosceles triangle perpendiculars are drawn to the two equal sides, their difference is equal to a perpendicular drawn from either extremity of the base to the opposite side.



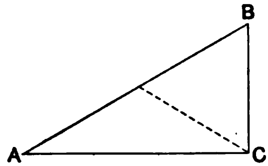
33. In the triangle *ABC*, *AE* and *CE* are the bisectors of $\angle A$ and the exterior angle *BCD* respectively.

Prove $\angle E = \frac{1}{2}\angle B$.



34. If one acute angle of a right-angled triangle is double the other, the hypotenuse is double the shorter leg.

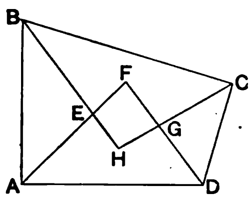
[See Exercise 17.]



35. Construct an equilateral triangle, having given its altitude.

36. The quadrilateral formed by the bisectors of the angles of a quadrilateral has its opposite angles supplementary.

[See Exercise 13.]



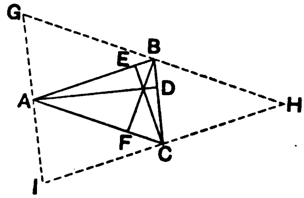
37. If the quadrilateral *ABCD* (see figure of Ex. 36) is a parallelogram, *EFGH* is a rectangle.

38. If the quadrilateral *ABCD* (see figure of Ex. 36) is a rectangle, *EFGH* is a square.

39. The bisectors of the *exterior* angles of a quadrilateral form a second quadrilateral whose opposite angles are supplementary.

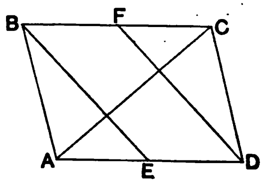
40. The altitudes of a triangle meet in a common point.

Suggestion. Through the three vertices of the △ *ABC* draw parallels to the opposite sides, forming △ *GHI*. Show that the altitudes of △ *ABC* are ⊥s to the sides of △ *GHI*, at their middle points.

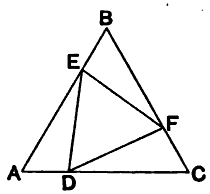


41. If the number of sides of an equiangular polygon is increased by four, each angle is increased by $\frac{1}{6}$ of a right angle. How many sides has the polygon? [§ 158.]

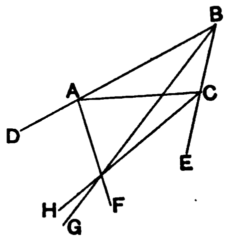
42. In the parallelogram *ABCD*, *BE* bisects *AD* and *DF* bisects *BC*. Prove that *BE* and *DF* trisect the diagonal *AC*.
[§ 239.]



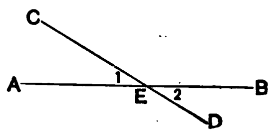
43. In the equilateral triangle *ABC*, the distances *AD*, *CF*, and *BE* are equal. Prove the triangle *DEF* equilateral.



44. *AF* and *HC* bisect the exterior angles *DAC* and *ACE*, and *BG* bisects the interior angle *B* of the triangle *ABC*. Prove that *AF*, *CH*, and *BG* meet in a common point.
[See § 233.]



45. If two lines that are on opposite sides of a third line meet at a point of that third line, making the non-adjacent angles equal, the two lines form one and the same line.

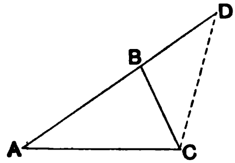


46. What is the greatest number of acute angles a convex polygon can have?

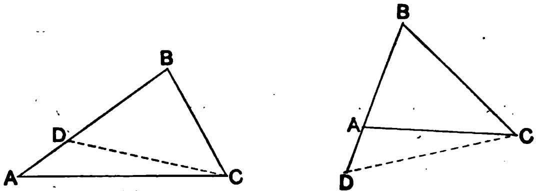
Suggestion. Show that if there were more than three acute angles the sum of the exterior angles of the polygon would exceed 4 R.A.'s.

47. Given two lines that would meet if sufficiently produced, draw the bisector of their angle, without prolonging the lines.

48. Construct a triangle, having given one angle, one of its including sides, and the sum of the other two sides.



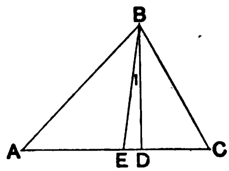
49. Construct a triangle, having given one angle, one of its including sides, and the difference of the other two sides.



(1) (2)

The side opposite the given angle may be less than the other unknown side (see Fig. 1), or it may be greater than the other unknown side (see Fig. 2).

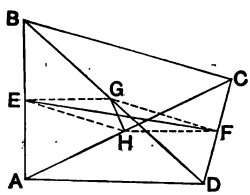
50. BE is the bisector of $\angle ABC$, and BD is an altitude of the triangle ABC. Prove that $\angle 1$ is one half the difference between the base angles A and C.



51. Through a point draw a line that shall be equally distant from two given points. [Two ways.]

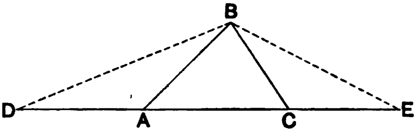
52. The line joining the middle points of two opposite sides of a quadrilateral bisects the line joining the middle points of the diagonals.

Suggestion. Prove that $EGFH$ is a parallelogram.



53. Of all triangles having the same base and equal altitudes the isosceles triangle has the least perimeter. [See **Ex. 20.**]

54. Construct a triangle, having given the perimeter and the two base angles.



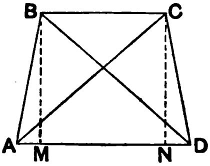
55. Construct a triangle, having given the lengths of the three medians. [§§ 244 and 245.]

56. If the diagonals of a trapezoid are equal, the non-parallel sides are equal.

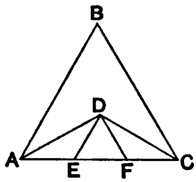
BM and *CN* are each ⊥ to *AD*.

Prove $\triangle ACN = \triangle DBM$,

and $\triangle ABM = \triangle DCN$.

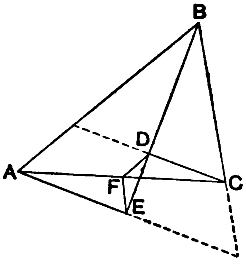


57. In the equilateral triangle *ABC*, *AD* and *DC* bisect the angles at *A* and *C*. *DE* is drawn ∥ to *AB*, and *DF* ∥ to *BC*. Prove that *AC* is trisected.



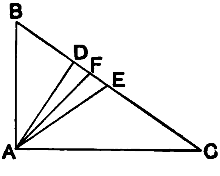
58. *AE* and *CD* are perpendiculars drawn from the extremities of *AC* to the bisector of ∠*B*. *FD* and *FE* join the feet of these perpendiculars with the middle point of *AC*.

Prove $FD = FE = \frac{1}{2}(AB - BC)$.



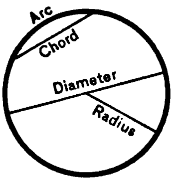
59. *ABC* is a R.A. △, *AD* is perpendicular to *BC*, and *AE* is the median to *BC*. *AF* bisects angle *DAE*.

Prove that *AF* also bisects angle *BAC*.



BOOK II

247. Definitions. A *circle* is a portion of a plane bounded by a curved line, all the points of which are equally distant from a point within called the center.



The bounding line is called the *circumference.*

A straight line from the center to any point in the circumference is a *radius.* It follows from the definition of circle that *all radii of the same circle are equal.*

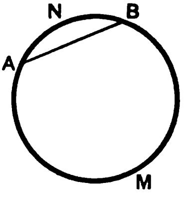
A straight line passing through the center and limited by the circumference is a *diameter.*

Every diameter is composed of two radii; therefore *all diameters of the same circle are equal.*

An *arc* is any portion of a circumference.

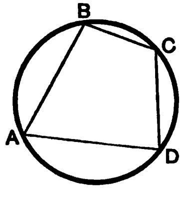
A *chord* is a straight line joining the extremities of an arc.

A chord is said to *subtend* the arc whose extremities it joins, and the arc is said to be *subtended by* the chord.



Every chord subtends two different arcs; thus the chord *AB* subtends the arc *ANB*, and also the arc *AMB*. Unless the contrary is specially stated, we shall assume the chord to belong to the smaller arc.

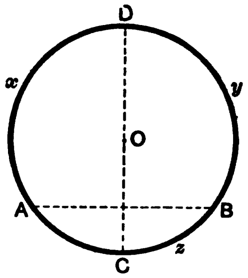
An *inscribed polygon* is a polygon whose vertices are in the circumference and whose sides are chords.



[The polygon *ABCD* is *inscribed in* the circle; the circle is also said to be *circumscribed about* the polygon.]

Proposition I. Problem

248. *To find the center of a given circle.*



Let *xyz* be the given circle.

Required to find its center.

Join any two points on the circumference, as A and B, by the line AB.

Bisect AB by the perpendicular DC.

Bisect DC.

Then is O the center of the circle.

By definition, the center of the circle is equally distant from A and B.

By § 48 the center is on DC.

By definition the center of the circle is equally distant from D and C.

Since the center is *on* DC, and is also equally distant from D and C, it must be at the middle point of DC, that is, at O.

Therefore, O is the center of the circle *xyz*. Q.E.F.

249. Corollary. *A line that is perpendicular to a chord and bisects it, passes through the center of the circle.*

Note. It follows from § 249 that the only chords in a circle that can bisect each other are diameters.

250. EXERCISE. Describe a circumference passing through two given points.

How many different circumferences can be described passing through two given points?

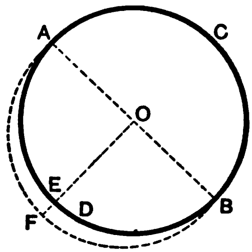
251. EXERCISE. Describe a circumference, with a given radius, and passing through two given points.

How many circumferences can be described in this case?

What limit is there to the length of the given radius?

PROPOSITION II. THEOREM

252. *A diameter divides a circle and also its circumference into two equal parts.*



Let AB be a diameter of the circle whose center is O.

To Prove that AB divides the circle and also its circumference into two equal parts.

Proof. Place ACB upon ADB so that AB is common.

Then will the curves ACB and ADB coincide, for if they do not there would be points in the two arcs unequally distant from the center, which contradicts the definition of circle.

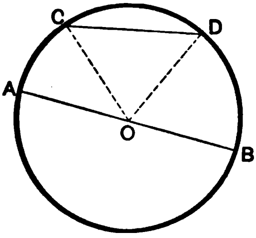
Therefore AB divides the circle and also its circumference into two equal parts. Q.E.D.

253. EXERCISE. Through a given point draw a line bisecting a given circle.

When can an infinite number of such lines be drawn?

PROPOSITION III. THEOREM

254. *A diameter of a circle is greater than any other chord.*



Let AB be a diameter of the ⊙ whose center is O, and CD be any other chord.

To Prove $AB > CD$.

Proof. Draw the radii OC and OD.

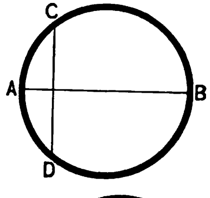
Apply § 168 to △ OCD, recollecting that $AB = OC + OD$.

Q.E.D.

255. EXERCISE. Prove this Proposition (§ 254), using a figure in which the given chord CD intersects the diameter AB.

256. EXERCISE. Through a point within a circle draw the longest possible chord.

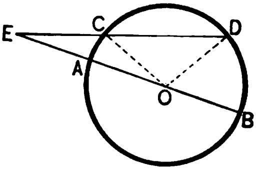
257. EXERCISE. The side AC of an inscribed triangle ABC is a diameter of the circle. Compare the angle B with angles A and C.



258. EXERCISE. AB is perpendicular to the chord CD, and bisects it.

Prove $AB > CD$.

259. EXERCISE. The diameter AB and the chord CD are prolonged until they meet at E.

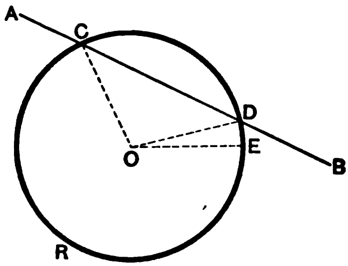


Prove $EA < EC$

and $EB > ED$.

Proposition IV. Theorem

260. ***A straight line cannot intersect a circumference in more than two points.***



Let *CDR* be a circumference and *AB* a line intersecting it at *C* and *D*.

To Prove that *AB* cannot intersect the circumference at any other point.

Proof. Suppose that *AB* did intersect the circumference in a third point *E*.

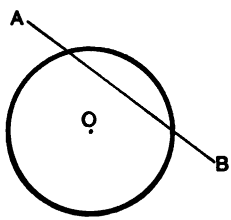
Draw the radii to the three points.

Now we have three equal lines (why equal?) drawn from the point *O* to the line *AB*, which contradicts (?).

Therefore the supposition that *AB* could intersect the circumference in more than two points is false. Q.E.D.

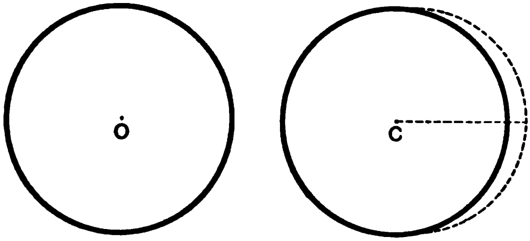
261. Exercise. Show by §§ 249 and 92 that *AB* cannot intersect the circumference in three points (*C*, *D*, and *E*).

262. Definition. A secant is a straight line that cuts a circumference.



PROPOSITION V. THEOREM

263. ***Circles having equal radii are equal; and conversely, equal circles have equal radii.***



Let the Ⓢ whose centers are O and C have equal radii.

To Prove the Ⓢ equal.

Proof. Place the ⊙ whose center is O upon the ⊙ whose center is C, so that their centers coincide.

Then will their circumferences also coincide, for if they do not, they would have unequal radii, which contradicts the hypothesis.

Since the circumferences coincide throughout, the circles are equal. Q.E.D.

CONVERSELY. **Let** the circles be equal.

To Prove that their radii are equal.

Proof. Since the circles are equal, they can be made to coincide.

Therefore their radii are equal. Q.E.D.

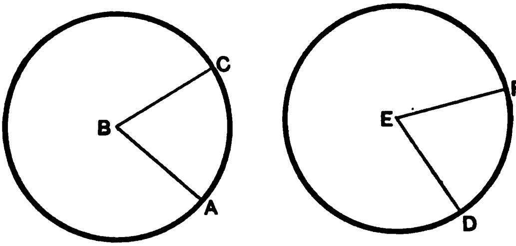
264. EXERCISE. Circles having equal diameters are equal; and conversely, equal circles have equal diameters.

265. EXERCISE. Two circles are described on the diagonals of a rectangle as diameters. How do the circles compare in size?

266. EXERCISE. If the circle described on the hypotenuse of a right-angled triangle as a diameter is equal to the circle described with one of the legs as a radius, prove that one of the acute angles of the triangle is double the other.

PROPOSITION VI. THEOREM

267. *In the same circle or in equal circles, radii forming equal angles at the center intercept equal arcs of the circumference; and conversely, radii intercepting equal arcs of the circumference form equal angles at the center.*



Let ABC and DEF be two equal angles at the centers of equal circles.

To Prove arc $CA =$ arc DF.

Proof. Place the circle whose center is B upon the circle whose center is E, so that $\angle B$ shall coincide with its equal $\angle E$.

Since the radii are equal, A will fall upon D and C upon F.

The arc AC will coincide with the arc DF. (Why?)

Therefore the arc $AC =$ arc DF. Q.E.D.

CONVERSELY. **Let** arc $CA =$ arc DF.

To Prove $\angle ABC = \angle DEF$.

Proof. Place the circle whose center is B upon the circle whose center is E, so that the circles coincide, and the arc AC coincides with its equal arc DF.

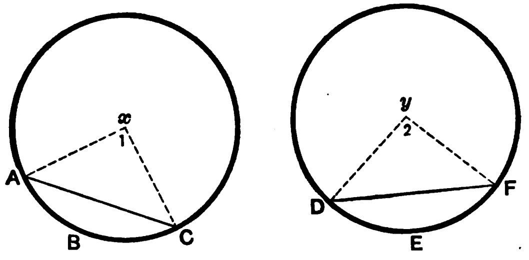
BC will then coincide with EF (?) and AB with DE. (?)

Consequently the angles ABC and DEF coincide and are equal. Q.E.D.

268. EXERCISE. Two intersecting diameters divide a circumference into four arcs which are equal, two and two.

Proposition VII. Theorem

269. *In the same circle, or in equal circles, if two arcs are equal, the chords that subtend them are also equal; and conversely, if two chords are equal, the arcs that are subtended by them are equal.*



Let ABC and DEF be two equal arcs in the equal ⓢ whose centers are x and y.

To Prove chord $AC =$ chord DF.

Proof. Draw the radii xA, xC, yD, and yF.

Show that $\angle 1 = \angle 2$.

Prove ⊿s AxC and DyF equal.

Whence $AC = DF$. Q.E.D.

Conversely. **Let** chord $AC =$ chord DF.

To Prove arc $ABC =$ arc DEF.

Proof. Draw the radii xA, xC, yD, and yF.

Prove ⊿s AxC and DyF equal.

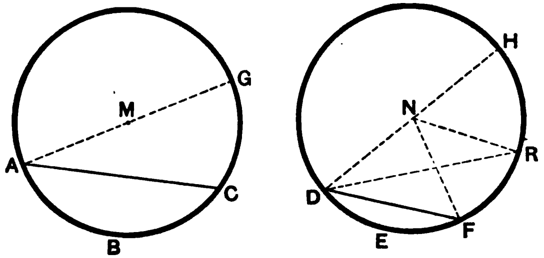
Whence $\angle 1 = \angle 2$.

$\therefore$ arc $ABC =$ arc DEF. (?) Q.E.D.

270. Exercise. If the circumference of a circle is divided into four equal parts and their extremities are joined by chords, the resulting quadrilateral is an equilateral parallelogram.

Proposition VIII. Theorem

271. *In the same circle, or in equal circles, if two arcs are unequal and each is less than a semi-circumference, the greater arc is subtended by the greater chord; and conversely, the greater chord subtends the greater arc.*



Let M and N be the centers of equal circles in which arc $ABC >$ arc DEF.

To Prove chord $AC >$ chord DF.

Proof. Draw the diameters AG and DH.

Place the semicircle ACG so that it shall coincide with the semicircle DFH, A falling on D and G on H.

Because the arc ABC is greater than the arc DEF, the point C will fall beyond F at some point R, the chord AC taking the position DR.

Draw the radii NF and NR.

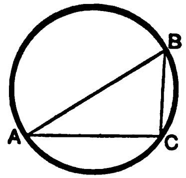
Apply § 181 to $\triangle$s DNF and DNR, proving

$$DR > DF. \quad \therefore AC > DF.$$ Q.E.D.

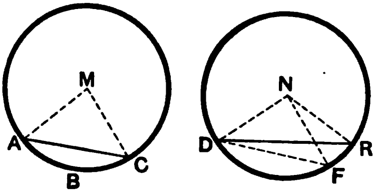
Conversely. **Let** chord $AC >$ chord DF.

To Prove arc $ABC >$ arc DEF.

Proof. Show that the arc ABC can neither be equal to the arc DEF nor less than it, $\therefore$ the arc ABC must be greater than the arc DEF. Q.E.D.



272. Exercise. *ABC* is a scalene triangle. How do the arcs *AB*, *BC*, and *AC* compare?

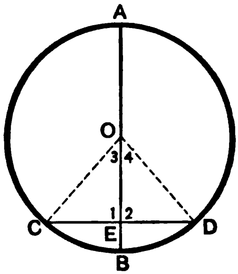


273. Exercise. Give a direct proof for the converse of Prop. VIII.

[Draw the radii and show that ∠*AMC* is less than ∠*RND*. Then place one circle upon the other, etc.]

Proposition IX. Theorem

274. *A diameter that is perpendicular to a chord bisects the chord and also the arc subtended by it.*



Let *AB* be a diameter ⊥ to *CD*.

To Prove $CE = ED$ and arc $CB =$ arc BD.

Proof. Draw the radii *OC* and *OD*.

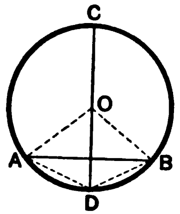
Prove △ *COE* and *OED* equal.

Whence $CE = ED$ and $\angle 3 = \angle 4$.

Show that arc $CB =$ arc BD. Q.E.D.

275. Corollary I. *The diameter AB also bisects the arc CAD.*

276. Corollary II. *Prove the six propositions that can be formulated from the following data, using any two for the hypothesis and the remaining two for the conclusion.*



A line that

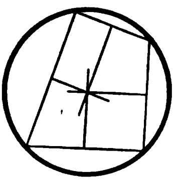
1. Passes through the center of the ⊙.
2. Bisects the chord.
3. Is perpendicular to the chord.
4. Bisects the arc.

[Prop. IX. itself is one of the six propositions, and is formed by using 1 and 3 as hypothesis, and 2 and 4 as conclusion; and the statement of § 249 uses 2 and 3 for its hypothesis and 1 for its conclusion.]

277. Corollary III. *Bisect a given arc.*

278. Exercise. What is the locus of the centers of parallel chords in a circle?

279. Exercise. Perpendiculars erected at the middle points of the sides of a quadrilateral inscribed in a circle pass through a common point. Is this true for inscribed polygons of more than four sides?



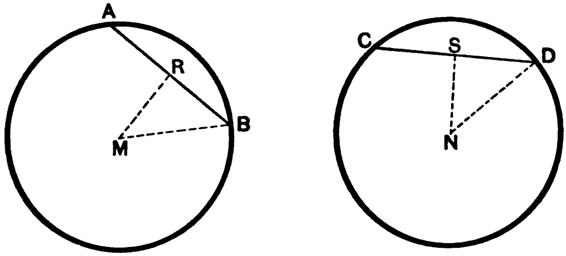
280. Exercise. Through a given point in a circle draw a chord that shall be bisected at the point.

281. Exercise. If the line joining the middle points of two chords in a circle passes through the center of the circle, prove that the chords are parallel.

282. Exercise. The chord *AB* divides the circumference into two arcs *ACB* and *ADB*. (See figure of § 276.) If *CD* is drawn connecting the middle points of these arcs, prove that it is perpendicular to *AB* and bisects it.

Proposition X. Theorem

283. *In the same circle or in equal circles equal chords are equally distant from the center; and conversely, chords that are equally distant from the center are equal.*



Let AB and CD be equal chords in the equal circles whose centers are M and N.

To Prove AB and CD equally distant from the centers.

Proof. Draw MR and NS ⊥ to AB and CD respectively.

MR and NS measure the distance of the chords from the centers. (§ 223.)

Draw the radii MB and ND.

Prove the $\triangle$ MRB and NSD equal.

Whence $MR = NS$. Q.E.D.

Conversely. **Let** AB and CD be equally distant from the centers ($MR = NS$).

To Prove $AB = CD$.

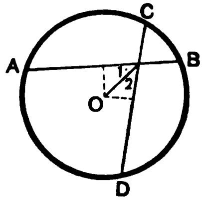
Proof. Prove $\triangle$ MRB and NSD equal.

Whence $RB = SD$.

Therefore $AB = CD$. (?) Q.E.D.

284. Exercise. What is the locus of the centers of equal chords in a circle?

285. Exercise. AB and CD are two intersecting chords, and they make equal angles with the line joining their point of intersection with the center of the circle. How do AB and CD compare in length?



286. Exercise. If two equal chords intersect in a circle, the segments of one chord are equal respectively to those of the other.

287. Exercise. If from a point without a circle two secants are drawn terminating in the concave arc, and if the line joining the center of the circle with the given point bisects the angle formed by the secants, the secants are equal.

288. Exercise. If two chords intersect in a circle and a segment of one of them is equal to a segment of the other, the chords are equal.

289. Exercise. The line joining the center of a circle with the point of intersection of two equal chords, bisects the angle formed by the chords.

290. Exercise. Through a given point of a chord to draw another chord equal to the given chord.

[*Suggestion.* — Apply § 285.]

291. Exercise. Through a given point in a circle only two equal chords can be drawn.

For what point in the circle is this statement untrue?

292. Exercise. If two equal chords be prolonged until they meet at a point without the circle, the secants formed are equal.

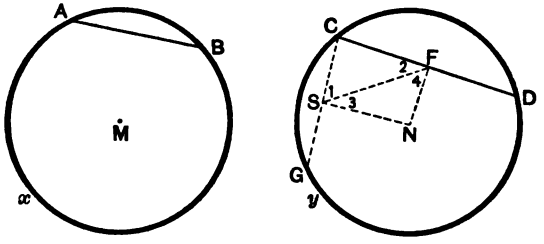
293. Exercise. Given three points A, B, and C on a circumference, to determine a fourth point X on that circumference, such, that if the chords AB and CX be prolonged until they meet at a point without the circle, the secants formed are equal.

294. Exercise. An inscribed quadrilateral $ABCD$ has its sides AB and CD parallel, and angles D and C equal.

Prove that the sides AD and BC are equally distant from the center of the circle.

PROPOSITION XI. THEOREM

295. *In the same circle or in equal circles, the smaller of two unequal chords is at the greater distance from the center; and conversely, if two chords are unequally distant from the center, the one at the greater distance is the smaller.*



Let M and N be the centers of equal ⓢ, and let $AB < CD$.

To Prove that AB is at a greater distance from M than CD is from N.

Proof. Place ⊙ xAB so that it coincides with ⊙ yCD, B falling on C and the chord AB taking the position CG.

Draw NS and NF ⊥ to GC and CD respectively. Draw SF.

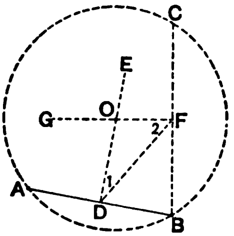
Prove	$\angle 1 > \angle 2$.		
Whence	$\angle 3 < \angle 4$.	(?)	
Whence	$NS > NF$.	(?)	Q.E.D.
CONVERSELY. **Let**	$NS > NF$.		
To Prove	$GC < CD$.		
Proof.	$\angle 3 < \angle 4$.	(?)	
	$\angle 1 > \angle 2$.	(?)	
	$CF > SC$.	(?)	
	$CD > GC$.	(?)	Q.E.D.

296. EXERCISE. Prove the converse to Prop. XI. *indirectly.* [Show that AB can neither be equal to nor greater than CD.]

297. Exercise. **Through a point within a circle draw the smallest possible chord.**

Proposition XII. Theorem

298. *Through three points not in the same straight line, one circumference, and only one, can be passed.*



Let A, B, and C be three points not in the same straight line.

To Prove that a circumference, and only one, can be passed through A, B, and C.

Proof. Draw AB and BC.

Bisect AB and BC by the ⊥s DE and FG.

Draw DF.

Show that $\angle 1 + \angle 2 < 2$ R.A.'s.

Whence DE and FG meet. (?)

O is equally distant from A and B. (?)

O is equally distant from B and C. (?)

Therefore O is equally distant from A, B, and C.

Therefore a circumference described with O as a center, and with OA, OB, or OC as a radius, will pass through A, B, and C.

The line DE contains *all* the points that are equally distant from A and B. (?)

The line GF contains *all* the points that are equally distant from B and C. (?)

Therefore their point of intersection is the *only* point that is equally distant from A, B, and C.

Therefore *only one* circumference can be passed through A, B, and C. Q.E.D.

299. Corollary. *Two circumferences can intersect in only two points.*

300. Exercise. Why cannot a circumference be passed through three points that are in a straight line?

301. Exercise. Circumscribe a circle about a given triangle.

302. Exercise. Show, by using §§ 298 and 249, that the perpendiculars erected to the sides of a triangle at their middle points pass through a common point.

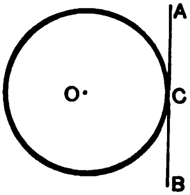
303. Exercise. Find the center of a given circle by using § 298.

304. Exercise. From a given point without a circle only two equal secants, terminating in the circumference, can be drawn.

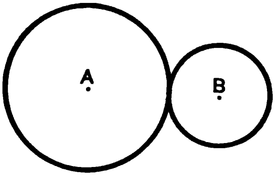
Suggestion. — Suppose that three equal secants could be drawn. Using the given point as a center and the length of the secant as a radius, describe a circle. Apply § 299.

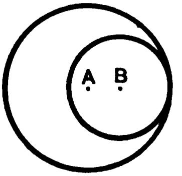
305. Exercise. Circumscribe a circle about a right-angled triangle. Show that the center of the circle lies on the hypotenuse.

306. Definitions. A straight line is *tangent* to a circle when it touches the circumference at one point only. The point at which the straight line meets the circumference is called the *point of tangency*. All other points of the straight line lie without the circumference. The circle is also said to be tangent to the line.



Two circles are tangent to each other when their circumferences touch at one point only. If one circle lies outside of the

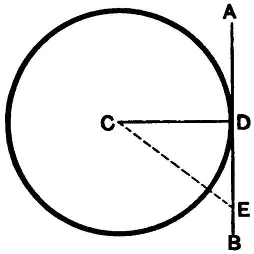




other, they are *tangent externally;* if one circle is within the other, they are *tangent internally.*

Proposition XIII. Theorem

307. *If a line is perpendicular to a radius at its outer extremity it is tangent to the circle at that point; and conversely, a tangent to a circle is perpendicular to the radius drawn to the point of tangency.*



Let AB be $\perp$ to the radius CD at D.

To Prove AB tangent to the circle.

Proof. Connect C with any other point of AB as E.

$$CE > CD. \quad (?)$$

Since CE is longer than a radius, E lies without the circumference.

E is *any* point on AB (except D).

Therefore *every* point on AB (except D) lies without the circumference, and AB touches the circumference at D only.

Q.E.D.

Conversely. **Let** AB be tangent to the $\odot$ at D.

To Prove $AB \perp$ to CD.

Proof. Connect C with any other point of AB as E.

Since AB is tangent to the circle at D, E lies without the circumference.

$$CE > CD. \quad (?)$$

CE is the distance from C to any point of AB (except D).

CD is therefore the shortest distance from C to AB.

$\therefore$ CD is perpendicular to AB. Q.E.D.

Corollary I. *At a given point on a circumference draw a tangent to the circle.*

Corollary II. *At a point on a circumference only one tangent can be drawn to the circle.*

308. Exercise. A perpendicular erected to a tangent at the point of tangency will pass through the center of the circle.

309. Exercise. If two tangents are drawn to a circle at the extremities of a diameter, they are parallel.

310. Exercise. The line joining the points of tangency of two parallel tangents passes through the center of the circle.

311. Exercise. If two unequal circles have the same center, a line that is tangent to the inner circle, and is a chord of the outer, is bisected at the point of tangency.

312. Exercise. Draw a line tangent to a circle and parallel to a given line.

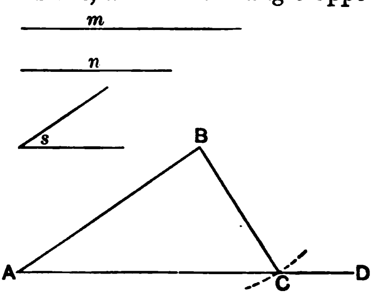
313. Exercise. Draw a line tangent to a circle and perpendicular to a given line.

314. Exercise. If an equilateral polygon is inscribed in a circle, prove that a second circle can be inscribed in the polygon.

315. Exercise. Circumscribe about a given circle a triangle whose sides are parallel to the sides of a given triangle.

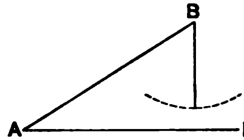
316. Exercise. To construct a triangle having given two sides and an angle opposite one of them.

Let m and n be the two given sides, and $\angle s$ the angle opposite side n.

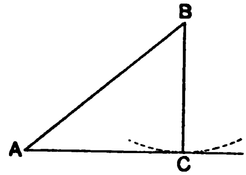


Required to construct the $\triangle$.

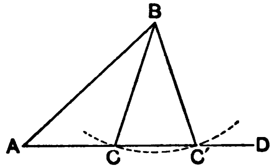
Lay off an indefinite line AD. At A construct $\angle A = \angle s$. Make $AB = m$. With B as a center, and n as a radius, describe an arc intersecting AD at C. Draw BC. Show that $\triangle ABC$ is the required $\triangle$.



SCHOLIUM. When the given angle is acute, and the side opposite the given angle is less than the perpendicular from *B* to *AD*, there is no construction.

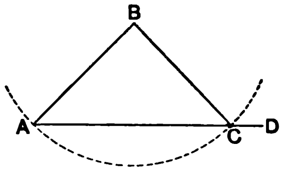


When the given angle is acute, and the side opposite the given angle is equal to the perpendicular from *B* to *AD*, there is one construction, and the △ is right-angled.



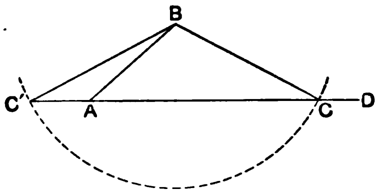
When the given angle is acute, and the side opposite the given angle is greater than the perpendicular from *B* to *AD* and is less than *AB*, there are two constructions.

Both △ *ABC* and △ *ABC′* fulfill the required conditions.



When the given angle is acute, and the side opposite the given angle is equal to *AB*, there is one construction.

When the given angle is acute, and the side opposite the given angle is greater than *AB*, there is one construction.



△ *ABC* fulfills the required conditions, but △ *ABC′* does not.

If the given angle is obtuse, the opposite side must be greater than *AB* (?), and there never can be more than one construction.

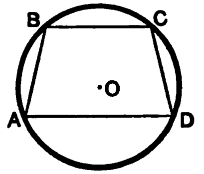
317. EXERCISE. Construct a triangle ABC in which $AB = 5$ inches, $\angle A = \frac{1}{3} RA$, and side $BC = 1, 2, 3, 4$, and 5 inches in turn.

State the number of solutions in each case.

How long must BC be in order to form a right-angled triangle?

319. Exercise. *ABCD* is a trapezoid inscribed in the circle whose center is *O*.

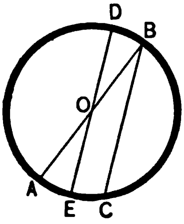
Prove that the non-parallel sides *AB* and *CD* are equal.



320. Exercise. Prove the converse of the preceding exercise, *i.e.* if two opposite sides of an inscribed quadrilateral are equal and not parallel, the quadrilateral is a trapezoid.

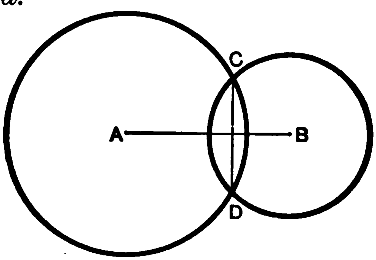
321. Exercise. The diagonals of an inscribed trapezoid are equal.

322. Exercise. The side *AB* of the inscribed angle *ABC* is a diameter. Prove that the diameter *DE* drawn parallel to *BC* bisects the arc *AC*.



Proposition XV. Theorem

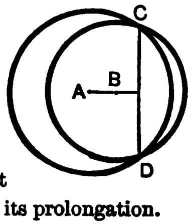
323. *If two circumferences intersect each other, the line joining their centers bisects at right angles their common chord.*



Let *AB* be the line joining the centers of two circumferences intersecting at *C* and *D*.

To Prove *AB* bisects *CD* at right angles.

Proof. Use § 49.

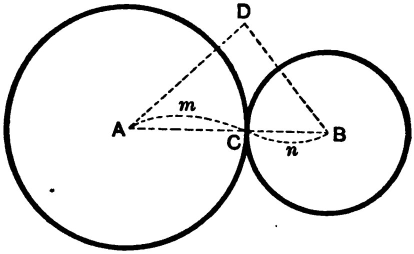


324. Exercise. Prove § 323, using this figure.

325. Exercise. The centers of all circles that pass through *C* and *D* (figure of § 323) are on *AB* or its prolongation.

PROPOSITION XVI. THEOREM

326. *If two circles are tangent, either externally or internally, their centers and the point of tangency are in the same straight line.*



Let A and B be the centers of two ⓢ tangent externally at C.

To Prove that A, C, and B are in the same straight line.

Proof. Draw the radii AC and BC to the point of tangency.

It is required to prove that ACB is a straight line.

If it can be shown that ACB is shorter than any other line joining A and B, then, by Axiom 14, ACB is a straight line.

I. To show that ACB is shorter than any other line joining A and B and passing through C.

Let $AmnB$ be any other line joining A and B and passing through C. $$AC + CB < AmC + CnB. \quad (?)$$

or $$ACB < AmnB.$$

II. To show that ACB is shorter than any line joining A and B and *not* passing through C.

Join A and B by any line ADB not passing through C.

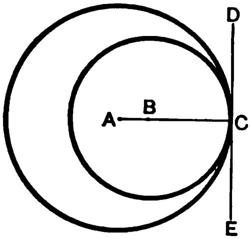
Since the circles touch at C only, any line joining the centers and not passing through C must pass outside of the circles, and must be greater than the sum of the radii.

$$\therefore ACB < ADB.$$

ACB is the shortest distance from A to B.

$\therefore$ ACB is a straight line. Q.E.D.

Let *A* and *B* be the centers of two circles tangent internally at *C*.



To Prove that *A*, *B*, and *C* are in a straight line.

Proof. At *C* draw *DE* tangent to the outer circle. (?)

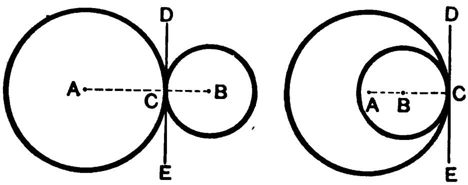
All the points of *DE* except *C* lie entirely without the outer circle, and consequently entirely without the inner circle.

DE touches the inner circle at *C* only, and is tangent to it also.

Draw the radii *AC* and *BC* to the point of tangency.

AC and *BC* are each ⊥ to *DE*. (?)

A, *B*, and *C* are in a straight line. (?) Q.E.D.



327. Corollary. *If two circles are tangent, either externally or internally, and if at their point of tangency a line is drawn tangent to one of the circles, it is tangent to the other also.*

328. Exercise. Two circles are tangent, and the distance between their centers is 10 in. The radius of one circle is 4 in. What is the radius of the other? (Two solutions.)

329. Exercise. Draw a common tangent to two circles tangent to each other. (§ 327.)

How many common tangents can be drawn to two circles that are tangent internally? Tangent externally? [In the latter case the student is expected at present to draw only one of the three common tangents.]

Proposition XVII. Theorem

330. *a. If two circles are entirely without each other and are not tangent, the distance between their centers is greater than the sum of their radii.*

b. If two circles are tangent externally, the distance between their centers is equal to the sum of their radii.

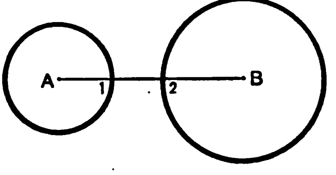
c. If two circles intersect, the distance between their centers is less than the sum and greater than the difference of their radii.

d. If two circles are tangent internally, the distance between their centers is equal to the difference of their radii.

e. If one circle lies wholly within another, and is not tangent to it, the distance between their centers is less than the difference of their radii.

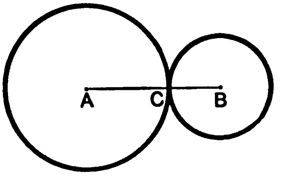
a

$AB >$ sum of radii. (?)



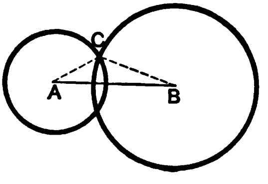
b

AB passes through C. (?)
$AB =$ sum of radii. (?)

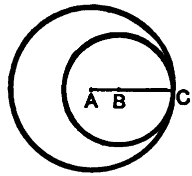


c

Draw the radii AC and BC.
$AB <$ sum of radii. (?)
$AB >$ difference of radii. (?)

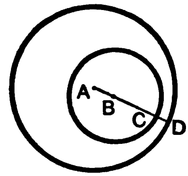


d



AB prolonged passes through C. (?)
AB = difference of radii. (?)

e



AD is the radius of the large ⊙.
BC is the radius of the small ⊙.
What is the difference of the radii?
$AB <$ difference of radii. (?)

[If two circles are concentric (*i.e.* have the same center) the distance between their centers is, of course, zero. This position manifestly comes under Case *e*.]

331. Corollary. *State and prove the converse of each case of Prop. XVII.* [Indirect proof.]

332. Exercise. If the centers of two circles are on a certain line, and their circumferences pass through a point of that line, the circles are tangent to each other.

333. Exercise. Two circles whose radii are 6 in. and 8 in. respectively, intersect. Between what limits does the length of the line joining their centers lie?

334. Exercise. With a given radius describe a circle tangent to a given circle at a given point. [Two solutions.]

335. Exercise. What is the locus of the centers of circles having a given radius and tangent to a given circle?

336. Exercise. Describe a circle having a given radius and tangent to two given circles.

Draw the figures for the next three constructions accurately and to scale. [1 ft. = $\frac{1}{2}$ in.]

337. Exercise. A and B are the centers of two circles. $AB = 7$ ft., radius of ⊙ $A = 2$ ft., and radius of ⊙ $B = 3$ ft. Describe a circle, with radius $2\frac{1}{2}$ ft., tangent to both.

338. Exercise. A and B are the centers of two circles. $AB = 1\frac{1}{2}$ ft., radius of $\odot A = 5$ ft., and radius of $\odot B = 2\frac{1}{2}$ ft. Describe a circle, with radius $1\frac{1}{2}$ ft., tangent to both.

339. Exercise. Describe three circles, with radii 1 ft., 2 ft., and 3 ft. respectively, and each tangent externally to both of the others.

340. Definition. The *ratio* of one quantity to another of the same kind is the quotient obtained by dividing the numerical measure of the first by the numerical measure of the second.

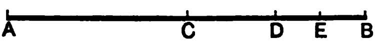
The ratio of 5 ft. to 7 ft. is $\frac{5}{7}$. The ratio of 7 lb. to 4 lb. is $\frac{7}{4}$, or $1\frac{3}{4}$. The ratio of the diagonal of a square to a side is $\sqrt{2}$ (as will be shown).

It is necessary that the two quantities be of the *same kind;* thus, it is impossible to express the ratio of 5 ft. to 7 lb.

Definitions. A *constant* is a quantity whose value remains unchanged throughout the same discussion.

A *variable* is a quantity whose value may undergo an indefinite number of successive changes in the same discussion.

The *limit of a variable* is a constant, from which the variable may be made to differ by less than any assignable quantity, but which it can never equal.



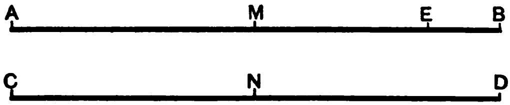
Suppose a point to move from A toward B, under the condition that in the first unit of time it shall pass over one half the distance from A to B; and in the next equal unit of time, one half of the remaining distance; and in each successive equal unit of time, one half the remaining distance.

It is plain that the point would never reach B, as there would always remain half of some distance to be covered.

The distance from A to the moving point is a variable, which is approaching the constant distance AB as a limit. The difference between the variable distance and the constant distance AB can be made less than any assignable quantity, but never can be made equal to zero.

Proposition XVIII. Theorem

341. *If two variables are always equal, and are each approaching a limit, their limits are equal.*



Let AM and CN be two variables that are always equal, and let AB and CD be their respective limits.

To Prove $AB = CD$.

Proof. Suppose AB and CD to be unequal, and $AB > CD$.

Lay off $AE = CD$.

Now, by the definition of limit, AM can be made to differ from AB by less than any assignable quantity, and therefore by less than EB.

So AM may be greater than AE.

By the definition of limit, $CN < CD$. But since $AE = CD$, $CN < AE$.

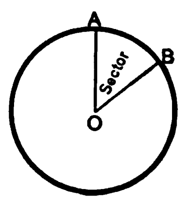
Now $AM > AE$ and $CN < AE$; but by hypothesis AM and CN are always equal.

The result being absurd, the supposition that AB and CD are unequal is false.

Therefore AB and CD are equal. Q.E.D.

342. Definition. Two magnitudes are *commensurable* when they have a common unit of measure; *i.e.* when they each contain a third magnitude a whole number of times.

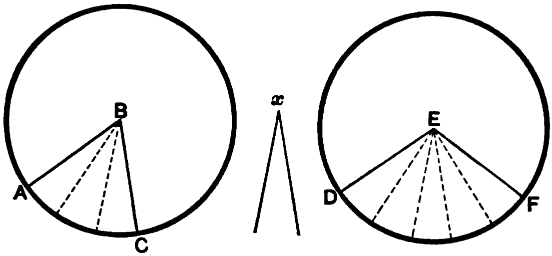
Two magnitudes are *incommensurable* when they have no common unit of measure; *i.e.* when there exists no third magnitude, however small, that is contained in each a whole number of times.



343. Definition. A *sector* is that part of a circle included between two radii and their intercepted arc.

Proposition XIX. Theorem

344. ***In the same circle or in equal circles, two angles at the center have the same ratio as their intercepted arcs.***



Case I

When the angles are commensurable.

Let ABC and DEF be commensurable angles at the centers of equal ⓢ.

To Prove $$\frac{\angle ABC}{\angle DEF} = \frac{AC}{DF}.$$

Proof. Since ∡ ABC and DEF are commensurable, they have a common unit of measure.

Let $\angle x$ be this unit, and suppose it is contained in $\angle ABC$ m times, and in $\angle DEF$ n times.

Whence $$\frac{\angle ABC}{\angle DEF} = \frac{m}{n}. \quad (1)$$

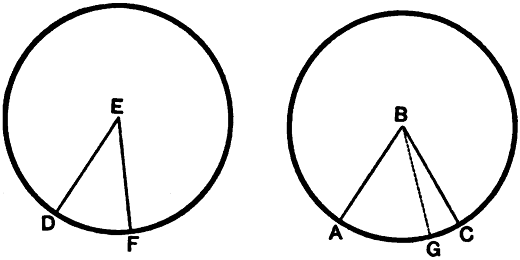
The small angles into which ∡ ABC and DEF are divided are equal, since each equals $\angle x$.

By § 267, the arcs into which AC and DF are divided by the radii are equal.

Since AC is composed of m of these equal arcs, and DF of n of these equal arcs, $$\frac{AC}{DF} = \frac{m}{n}. \quad (2)$$

Apply Axiom 1 to (1) and (2).

$$\frac{\angle ABC}{\angle DEF} = \frac{AC}{DF}.$$ Q.E.D



CASE II

When the angles are incommensurable.

Let ABC and DEF be two incommensurable angles at the centers of equal ⓢ.

To Prove $$\frac{\angle ABC}{\angle DEF} = \frac{AC}{DF}.$$

Proof. Let $\angle DEF$ be divided into a number of equal angles, and let one of these be applied to $\angle ABC$ as a unit of measure.

Since ∡ ABC and DEF are incommensurable, ABC will not contain this unit of measure exactly, but a certain number of these angles will extend as far as, say, ABG, leaving a remainder $\angle GBC$, smaller than the unit of measure.

Since ∡ ABG and DEF are commensurable, (?)

$$\frac{\angle ABG}{\angle DEF} = \frac{AG}{DF} \text{ by Case I.}$$

By increasing indefinitely the number of parts into which $\angle DEF$ is divided, the parts will become smaller and smaller, and the remainder $\angle GBC$ will also diminish indefinitely.

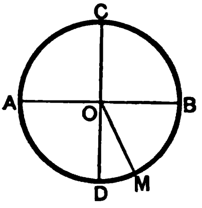
Now $\frac{\angle ABG}{\angle DEF}$ is evidently a variable, as is also $\frac{AG}{DF}$, and these variables are always equal to each other. (Case I.)

The limit of the variable $\frac{\angle ABG}{\angle DEF}$ is $\frac{\angle ABC}{\angle DEF}$.

The limit of the variable $\frac{AG}{DF}$ is $\frac{AC}{DF}$.

By § 341, $$\frac{\angle ABC}{\angle DEF} = \frac{AC}{DF}.$$ Q.E.D.

345. Corollary. *In the same circle, or in equal circles, sectors are to each other as their arcs.* [The proof is analogous to that of the Proposition, substituting *sector* for *angle.*]



346. Scholium. If two diameters are drawn perpendicular to each other, four right angles are formed at the center of the circle. By § 267, the circumference is divided into four equal arcs called quadrants.

If one of these right angles were divided into any number of equal parts, it could be shown by § 267, that the quadrant subtending the right angle is also divided into the same number of equal parts. If, for example, the right angle at the center were divided into four equal parts, the arcs intercepted by the sides of these angles would each be one fourth of a quadrant; and conversely, radii intercepting an arc that is one fourth of a quadrant, form an angle at the center which is one fourth of a right angle.

If any angle as $\angle DOM$ be taken at random and compared with a right angle,

By § 344, $$\frac{\angle DOM}{\text{R. A.}} = \frac{DM}{\text{quadrant}},$$

i.e. the angle DOM is the same part of a right angle that its intercepted arc is of a quadrant.

In this sense an angle at the center is said to be *measured by* its intercepted arc.

347. Scholium. A quadrant is usually conceived to be divided into ninety equal parts, each part called a *degree of arc.*

The angle at the center that is measured by a degree of arc is called a *degree of angle.*

The degree is divided into sixty equal parts called *minutes,* and each minute is again subdivided into sixty equal parts called *seconds.*

Degrees, minutes, and seconds are designated by the symbols $^\circ$, $'$, $''$ respectively. Thus. 49 degrees, 27 minutes, and 35 seconds, is written $49^\circ\ 27'\ 35''$.

348. Exercise. Add 23° 46′ 27″ and 19° 21′ 36″.

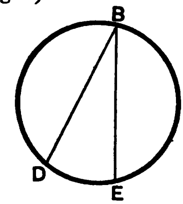
349. Exercise. Subtract 15° 42′ 39″ from 93° 16′ 25″.

350. Exercise. How many degrees in an angle of an equilateral triangle?

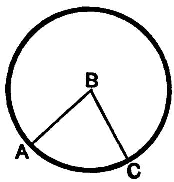
351. Exercise. Multiply 13° 27′ 35″ by 3, and add the product to one half of 12° 15′ 10″.

352. Exercise. How many degrees are there in each angle of an isosceles right-angled triangle?

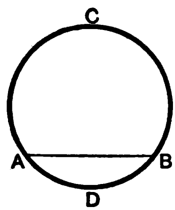
353. Exercise. Express in degrees, minutes, and seconds the value of one angle of a regular heptagon (a seven-sided polygon).



354. Definition. An *inscribed angle* is an angle whose vertex is in the circumference and whose sides are chords.

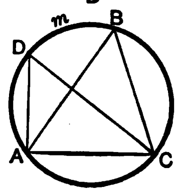


The symbol ~ is used for the phrase *is measured by*. Thus, $\angle ABC \sim$ arc AC is read: The angle ABC is measured by the arc AC.



A *segment* is that part of a circle which is included between an arc and its chord.

[ACB and ADB are both segments.]



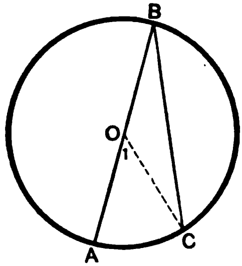
An angle is *inscribed in a segment* when its vertex is in the arc of the segment and its sides terminate in the extremities of that arc.

[$\angle ABC$ and $\angle ADC$ are inscribed in the segment AmC.]

PROPOSITION XX. THEOREM

355. ***An inscribed angle is measured by one half of the arc intercepted by its sides.***

CASE I



Let $\angle ABC$ be an inscribed angle having a diameter for one of its sides.

To Prove $\angle ABC \sim \frac{1}{2} AC$.

Proof. Draw the radius OC.

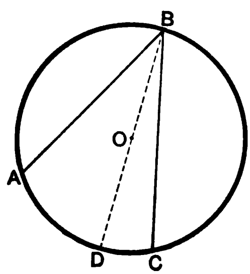
Prove $\angle 1 = 2 \angle B$.

$\angle 1 \sim AC$. (§ 346.)

$\therefore \angle B$, which is one half $\angle 1$, is measured by one half the arc AC. Q.E.D.

CASE II

Let $\angle ABC$ be an inscribed angle having the center between its sides.



To Prove $\angle ABC \sim \frac{1}{2} AC$.

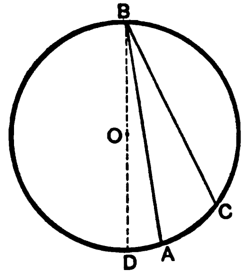
Draw the diameter BD.

$\angle ABD \sim \frac{1}{2} AD$. (Case I.)

$\angle DBC \sim \frac{1}{2} DC$. (Case I.)

$\angle ABC$, which is the sum of $\angle$s ABD and DBC, is measured by the sum of their measures $(\frac{1}{2} AD + \frac{1}{2} DC)$, that is, by $\frac{1}{2} AC$. Q.E.D.

Case III



Let $\angle ABC$ be an inscribed angle having the center without its sides.

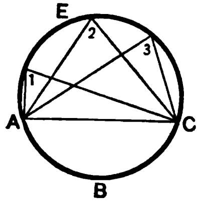
To Prove $\angle ABC \sim \frac{1}{2} AC$.

Proof. Draw the diameter BD.

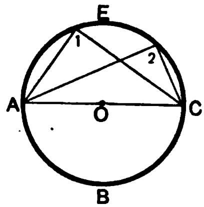
$\angle DBC \sim \frac{1}{2} DC$. (?)

$\angle DBA \sim \frac{1}{2} DA$. (?)

$\angle ABC$, which is the difference between $\angle$s DBC and DBA, is measured by the difference of their measures $(\frac{1}{2} DC - \frac{1}{2} DA)$, that is, by $\frac{1}{2} AC$. Q.E.D.

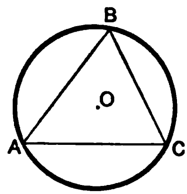


356. Corollary I. *Angles inscribed in the same segment are equal.*

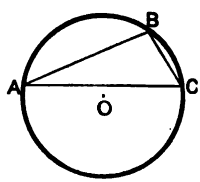


357. Corollary II. *Angles inscribed in a semicircle are right angles.*

[$\angle 1 \sim \frac{1}{2} ABC$. But $\frac{1}{2}$ of the arc ABC is a quadrant. Therefore, by § 346, $\angle 1$ is a right angle.]

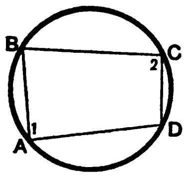


358. Corollary III. *An angle inscribed in a segment that is greater than a semicircle is acute.*



359. Corollary IV. *An angle inscribed in a segment that is less than a semicircle is obtuse.*

360. Corollary V. *The opposite angles of an inscribed quadrilateral are supplementary.*



[Show that the sum of the measures of ∠s 1 and 2 is a semicircumference, or two quadrants.]

361. Exercise. The sides of an inscribed angle intercept an arc of 50°. What is the size of the angle?

362. Exercise. How many degrees in an arc intercepted by the sides of an inscribed angle of 40°?

363. Exercise. If the opposite angles of a quadrilateral are supplementary, a circle may be circumscribed about it. (Converse of Cor. V.)

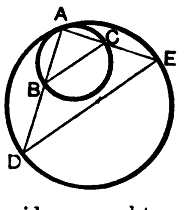
[Pass a circumference through three of the vertices. Then show that the fourth vertex can fall neither without nor within the circumference.]

364. Exercise. Show by § 355 that the sum of the angles of a triangle is two right angles.

365. Exercise. Any parallelogram inscribed in a circle is a rectangle.

366. Exercise. Two circles are tangent at *A*. *AD* and *AE* are drawn through the extremities of a diameter *BC*.

Prove that *DE* is also a diameter.



367. Exercise. Prove the preceding exercise when the two circles are tangent externally.

368. Exercise. The angles of an inscribed trapezoid are equal two and two.

369. Exercise. Prove § 355, Case I, using the figure of § 322.

370. Exercise. Two chords *AB* and *CD* intersect in a circle at the point *E*. Their extremities are joined by the lines *AC* and *DB*. Prove the ∆s *ACE* and *BDE* mutually equiangular.

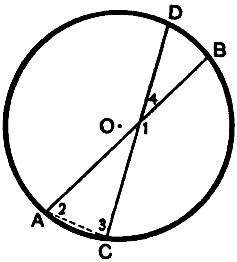
371. Exercise. The sum of one set of alternate angles of an inscribed octagon is equal to the sum of the other set.

Proposition XXI. Theorem

372. *An angle formed by two intersecting chords is measured by one half the sum of the arc intercepted by the sides of the angle and the arc intercepted by the sides of its vertical angle.*

Let $\angle 1$ be an angle formed by the intersecting chords AB and CD.

To Prove $\angle 1 \sim \frac{1}{2}(AD + BC)$.



Proof. Draw the chord AC.

$\angle 1 = \angle 2 + \angle 3$. (?)

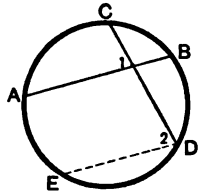
$\angle 2 \sim \frac{1}{2} BC$. (?)

$\angle 3 \sim \frac{1}{2} AD$. (?)

Since $\angle 1$ is the sum of $\angle$s 2 and 3, it is measured by the sum of their measures,

$\therefore \angle 1 \sim \frac{1}{2}(AD + BC)$. Q.E.D.

373. Exercise. Derive the measure of $\angle 4$ in the above figure.



374. Exercise. If in the above figure the arc BC contains 124° and the arc AD contains 172°, how many degrees in $\angle 1$?

375. Exercise. Prove $\angle 1 \sim \frac{1}{2}(AC + BD)$, using this figure.

[DE is drawn parallel to AB.]

376. Exercise. If angle 1 (figure § 375) contains 85° and arc BC contains 55°, how many degrees in the arc AD?

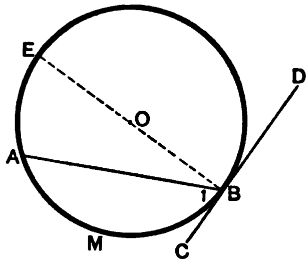
377. Exercise. Four points A, B, C, and D are so taken in a circumference that the arcs AB, BC, CD, and DA form a geometrical progression ($AB = 2\ BC$, $BC = 2\ CD$, etc.). Find the values of each of the angles formed by the intersection of the chords AC and BD.

PROPOSITION XXII. THEOREM

378. ***An angle formed by a chord meeting a tangent at the point of tangency is measured by one half the arc intercepted by its sides.***

Let $\angle 1$ be an angle formed by the chord AB and the tangent CD.

To Prove $\angle 1 \sim \frac{1}{2} AMB$.



Proof. Draw the diameter EB to the point of tangency.

$$\angle EBC = 1 \text{ R.A.} \quad (?)$$

A right angle is measured by a quadrant. (?)

$\frac{1}{2}$ arc EMB is a quadrant. (?)

$$\angle EBC \sim \frac{1}{2} EMB.$$

$$\angle EBA \sim \frac{1}{2} AE. \quad (?)$$

$\angle 1$, which is the difference between $\angle EBC$ and $\angle EBA$, is measured by the difference of their measures.

$$\angle 1 \sim \frac{1}{2} EMB - \frac{1}{2} EA.$$

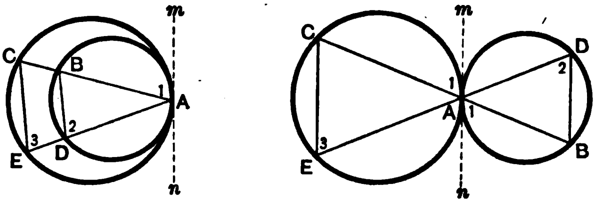
$$\angle 1 \sim \frac{1}{2} AMB. \qquad \text{Q.E.D.}$$

Similarly, it may be shown that $\angle ABD$, which is the sum of R.A. EBD and $\angle EBA$, is measured by the sum of their measures, which is $\frac{1}{2}$ arc AEB.

379. **EXERCISE.** **A chord that divides a circumference into arcs containing 80° and 280°, respectively, is met at one extremity by a tangent. What are the angles formed by the lines?**

380. **EXERCISE.** **A chord is met at one extremity by a tangent, making with it an angle of 55°. Into what arcs does the chord divide the circumference?**

381. Exercise. **If two circles are tangent either externally or internally, and through the point of contact two lines are drawn meeting one circumference in *B* and *D* and the other in *E* and *C*, *BD* and *EC* are parallel.**

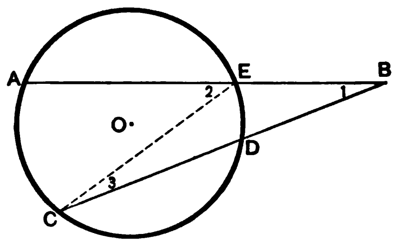


[Draw the common tangent *mn*. Show that $\angle 3$ and $\angle 2$ each equals $\angle 1$.]

382. Exercise. **If tangents be drawn to the two circles at the points *B* and *C* (see the figures of the preceding exercise), prove they are parallel.**

Proposition XXIII. Theorem

383. *An angle formed by two secants meeting without the circle is measured by one half the difference of the arcs intercepted by its sides.*



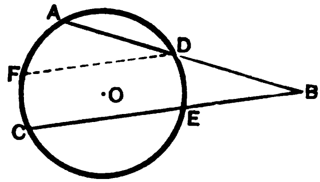
Let $\angle 1$ be an angle formed by the two secants *AB* and *CB*.

To Prove $\angle 1 \sim \frac{1}{2}(AC - DE)$.

Proof. Draw the chord *CE*.

$$\angle 1 = \angle 2 - \angle 3. \quad (?)$$

$\angle 1$ is therefore measured by the difference of the measures of $\angle$s 2 and 3, *i.e.* by $\frac{1}{2}(AC - DE)$. **Q.E.D.**

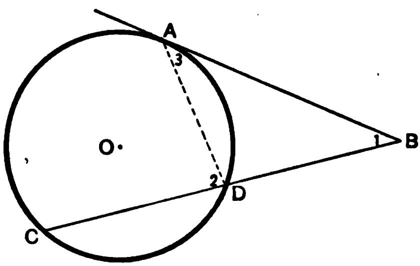


384. Exercise. If the secants AB and CB in the figure of § 383 intercept arcs of 70° and 42°, what is the size of $\angle B$?

385. Exercise. Prove § 383, using this figure. [DF is ∥ to BC.]

Proposition XXIV. Theorem

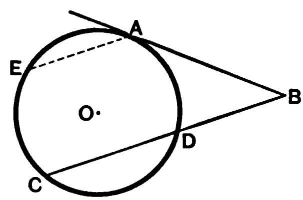
386. *An angle formed by a tangent and a secant meeting without the circle is measured by one half the difference of the arcs intercepted by its sides.*



Let $\angle 1$ be an angle formed by the tangent AB and the secant CB.

To Prove $\angle 1 \sim \frac{1}{2}(AC - AD)$.

Proof. Similar to that of § 383.



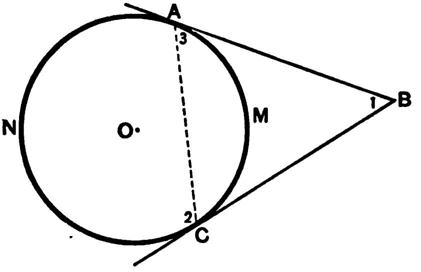
Exercise. Prove § 386, using this figure. [EA is ∥ to BC.]

387. Exercise. A tangent and a secant meeting without a circle form an angle of 35°. One of the arcs intercepted by them is 15°. How many degrees in the other?

388. A triangle ABC is inscribed in a circle. The angle B is equal to 50°, and the angle C is equal to 60°. What angle does a tangent at A make with BC produced to meet it?

PROPOSITION XXV. THEOREM

389. ***An angle formed by two tangents is measured by one half the difference of the arcs intercepted by its sides.***

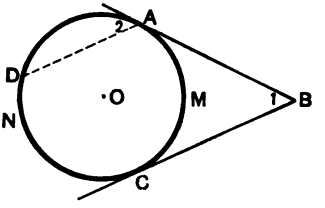


Let $\angle 1$ be an angle formed by the tangents AB and CB.

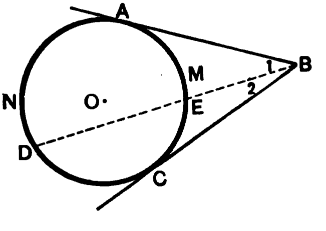
To Prove $\angle 1 \sim \frac{1}{2}(ANC - AMC)$.

Proof. Similar to that of §§ 383 and 386.

EXERCISE. Prove § 389, using this figure.
[AD is drawn parallel to BC.]



EXERCISE. Prove § 389, using this figure.
[BD is any secant drawn from B.]

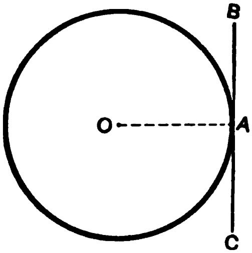


390. EXERCISE. The angle formed by two tangents is 74°. How many degrees in each of the two arcs intercepted by them?

PROPOSITION XXVI. PROBLEM

391. ***Through a given point to draw a tangent to a given circle.***

CASE I



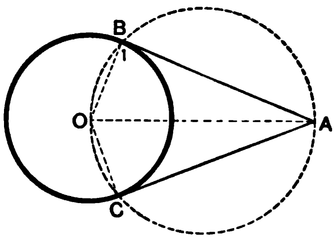
When the given point is on the circumference.

Let A be the given point on the circumference of the circle whose center is O.

Required to draw a tangent to the circle through A.

See § 307.

CASE II



When the given point is without the circumference.

Let A be the given point without the circle whose center is O.

Required to draw a tangent to the circle through A.

Draw OA.

On OA as a diameter, describe a circumference, cutting the given circumference at B and C.

Draw AB and AC.

AB and AC are the required tangents.

Draw the radii OB and OC.

$\angle 1$ is a right angle. (?)

AB is tangent to the circle. (?)

Similarly, AC is tangent to the circle. Q.E.F.

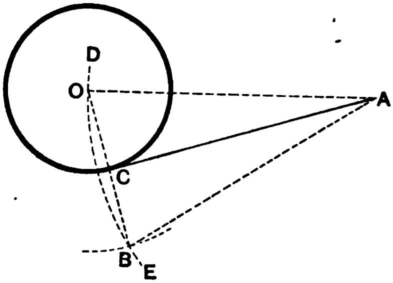
Case II. *Second Method*

392. With A as center and AO as a radius, describe the arc DE.

With O as a center and the diameter of the given circle as a radius, describe an arc cutting DE at B.

Draw OB intersecting the given circle at C.

Draw AC. Then AC is the required tangent.



[The proof is left for the student.]

393. Corollary. *The two tangents drawn from a point to a circle are equal; and the line joining the point with the center of the circle bisects the angle between the tangents, and also bisects the chord of contact* (BC *in the figure to first method*) *at right angles.*

394. Scholium. When the given point is without the circle, *two* tangents can be drawn; when it is on the circumference, *one*, and when it is within the circle, *none*.

395. Definition. A polygon is *circumscribed about a circle* when each of its sides is tangent to the circle. In this case the circle is said to be *inscribed in* the polygon.

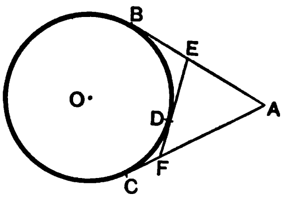
396. Exercise. If a quadrilateral is circumscribed about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.

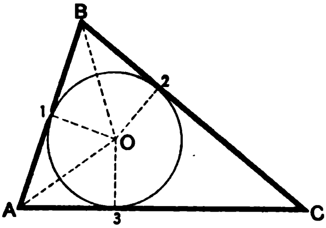
Suggestion. Use § 393.

397. Exercise. From the point A two tangents AB and AC are drawn to the circle whose center is O.

At any point D on the included arc BC, a third tangent FE is drawn.

Prove that the perimeter of the $\triangle AEF$ is constant, and equal to the sum of the tangents AB and AC.





398. Exercise. To inscribe a circle in a given triangle.

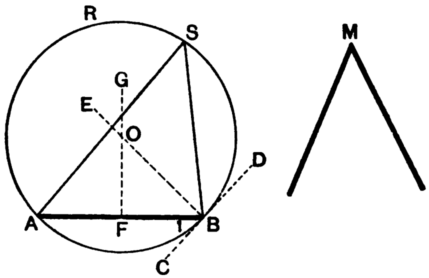
Bisect two of the angles. Show that their point of meeting is equally distant from the three sides.

∴ the three perpendiculars $O1$, $O2$, and $O3$ are equal.

With O as a center and with $O1$ as a radius, describe the required circle.

Proposition XXVII. Problem

399. *On a given line to construct a segment that shall contain a given angle.*



Let AB be the given line and $\angle M$ the given angle.

Required to construct on AB a segment that shall contain $\angle M$.

Draw CD through B, making $\angle 1 = \angle M$.

Erect $BE \perp$ to CD and bisect AB by the $\perp FG$.

Prove that BE and FG meet at some point O.

Show that O is equally distant from A and B.

With O as a center describe a circle passing through A and B.

DC is tangent to this circle. (?) $\angle 1 \sim \frac{1}{2} AB$. (?)

Inscribe any angle as $\angle ASB$ in the segment ARB.

$\angle ASB \sim \frac{1}{2} AB$. (?) $\angle ASB = \angle 1 = \angle M$. (?)

The segment ARB is the required segment, since any angle inscribed in it is equal to $\angle M$. Q.E.F.

400. Exercise. On a given line construct a segment that shall contain an angle of 135°.

401. Exercise. What is the locus of the vertices of the vertical angles of the triangles having a common base and equal vertical angles?

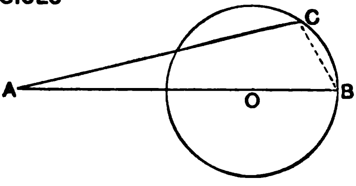
402. Exercise. Construct a triangle, having given the base, the vertical angle, and the altitude.

403. Exercise. Construct a triangle, having given the base, the vertical angle, and the medial line to the base.

EXERCISES

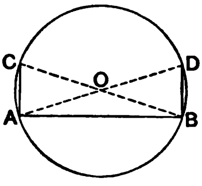
1. Two secants, AB and AC, are drawn to the circle, and AB passes through the center.

Prove $\angle ACB > \angle ABC$.



2. One angle of an inscribed triangle is 42°, and one of its sides subtends an arc of 110°.

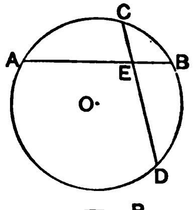
Find the angles of the triangle.



3. Two chords drawn perpendicular to a third chord at its extremities are equal. [Show that BC and AD are diameters, and that $\triangle ABC$ and ADB are equal.]

4. AB and CD are two chords intersecting at E, and $CE = BE$.

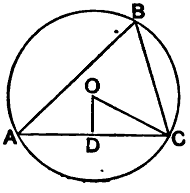
Prove arc AC = arc BD.



5. ABC is a triangle inscribed in the circle, whose center is O.

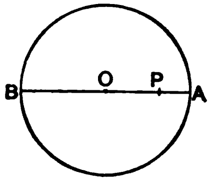
OD is drawn perpendicular to AC.

Prove $\angle DOC = \angle B$.

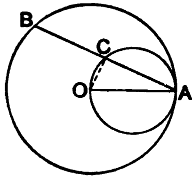


6. What is the locus of the centers of circles tangent to a line at a given point?

7. P is any point within the circle whose center is O. Prove that PA is the shortest line and PB the longest line from P to the circumference.



8. If a circle is described on the radius of another circle as a diameter, any chord of the greater circle drawn from the point of contact is bisected by the circumference of the smaller circle.



9. If from a point on a circumference a number of chords are drawn, find the locus of their middle points. (Ex. 8.)

10. From two points on opposite sides of a given line, draw two lines meeting in the given line, and making a given angle with each other. (§ 399.)

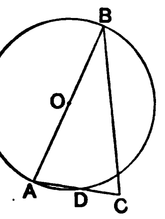
11. Work Ex. 10, taking the two points on the same side of the given line.

When is the problem impossible?

12. One of the equal sides of an isosceles triangle is the diameter of a circle.

Prove that the circumference bisects the base.

[Show that BD is $\perp$ to AC.]



13. What is the locus of the centers of circles having a given radius and tangent to a given line?

14. Describe a circle having a given radius and tangent to two non-parallel lines.

How many circles can be drawn?

15. Any parallelogram that can be circumscribed about a circle is equilateral.

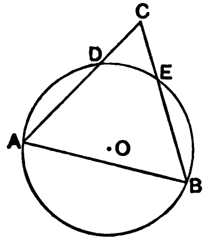
16. The bisector of an angle of an inscribed quadrilateral meets the bisector of the opposite exterior angle on the circumference.

17. Describe a circle having a given radius and tangent to a given line and also to a given circle.

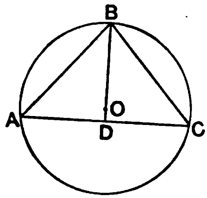
18. The base AB of the isosceles triangle ABC is a chord of a circle, the circumference of which intersects the two equal sides at D and E.

Prove $CD = CE$.

[$\angle A$ and $\angle B$ are measured by equal arcs.]

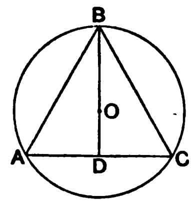


19. If an isosceles triangle is inscribed in a circle, prove that the bisector of the vertical angle passes through the center of the circle.

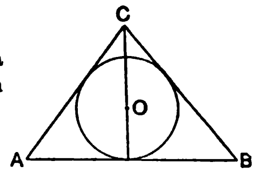


20. The altitude of an equilateral triangle is one and a half times the radius of the circumscribed circle.

[Use the preceding Exercise and § 245.]



21. If a triangle is circumscribed about a circle, the bisectors of its angles pass through the center of the circle. [§ 230.]



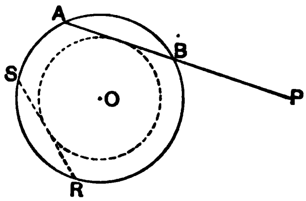
22. The altitude of an equilateral triangle is three times the radius of the inscribed circle.

[Use Ex. 21 and § 245.]

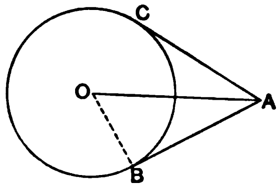
23. The angle between two tangents to a circle is 30°. Find the number of degrees in each of the intercepted arcs.

24. From a given point draw a line cutting a circle and making the chord equal to a given line.

[The chord *RS* is equal to the given line. The dotted circle is tangent to *RS*.]



25. Find the angle formed by two tangents to a circle, drawn from a point the distance of which from the center of the circle is equal to the diameter.

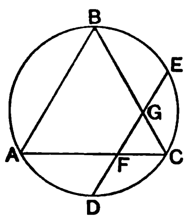


26. With a given radius describe a circle that shall pass through a given point and be tangent to a given line.

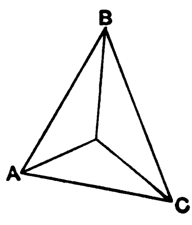
27. With a given radius describe a circle that shall pass through a given point and be tangent to a given circle.

28. From a point without a circle draw the shortest line to the circumference.

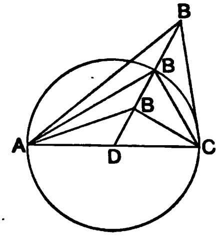
29. *ABC* is an inscribed equilateral triangle. *DE* joins the middle points of the arcs *BC* and *CA*. Prove that *DE* is trisected by the sides of the triangle.



30. Find a point within a triangle such that the angles formed by drawing lines from it to the three vertices of the triangle shall be equal to each other. (§ 399.)

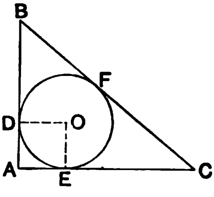


31. A median BD is drawn from angle B in the triangle ABC. Show that angle B is a right angle when BD is equal to one half of the base AC, an acute angle when BD is greater than one half of AC, and an obtuse angle when BD is less than one half of AC.



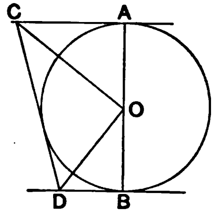
32. In any right-angled triangle, the sum of the two legs is equal to the sum of the hypotenuse and the diameter of the inscribed circle.

[Tangents drawn from a point to a ⊙ are equal.]



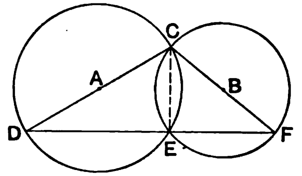
33. Tangents CA and DB drawn at the extremities of the diameter AB meet a third tangent CD at C and D. Draw CO and DO.

Prove $CD = CA + DB$ and $\angle COD = 1$ R.A.



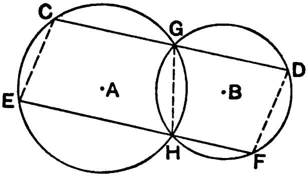
34. If from one point of intersection of two circles two diameters are drawn, the other extremities of the diameters and the other point of intersection of the circles are in a straight line.

[Draw DE and EF. Show that $\angle DEC + \angle CEF = 2$ R.A.'s.]



35. Through the points of intersection of two circles two parallel secants are drawn, terminating in the curves. Prove the secants equal.

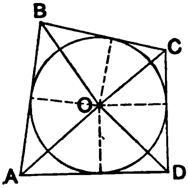
[Show that the quadrilateral $ECDF$ has its opposite angles equal, each to each.]



36. In a given circle draw a chord the length of which shall be twice its distance from the center.

37. Three equal circles are tangent to each other. Through their points of contact three common tangents are drawn.

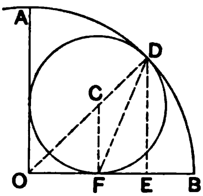
Prove. 1. The three tangents meet in a common point.
2. The point of meeting is equally distant from the three points of contact.



38. The sum of the angles subtended at the center of a circle by two opposite sides of a circumscribed quadrilateral is equal to two right angles.

[To prove $\angle AOB + \angle COD = 2$ R.A.'s.]

39. Find the locus of points such that tangents drawn from them to a given circle shall equal a given line.



40. Inscribe a circle in a given quadrant.

[OD bisects $\angle AOB$. DE is $\perp$ to OB. DF bisects $\angle ODE$.]

41. If the tangents to a circle at the four vertices of an inscribed rectangle (not a square) be prolonged, they form a rhombus.

42. From any point (not the center) within a circle only two equal straight lines can be drawn to the circumference.

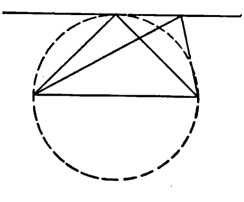
43. Given a circle and a point within or without (not the center), using the given point as a center to describe a circle, the circumference of which shall bisect the circumference of the given circle.

44. In a given circle inscribe a triangle, the angles of which are respectively equal to the angles of a given triangle.

[Draw a tangent to the ⊙, and from the point of contact draw two chords, making the three ∠s at the point of contact equal to the ∠s of the △.]

45. Circumscribe about a given circle a triangle, the angles of which are respectively equal to the angles of a given triangle.

[Inscribe a ⊙ in the given △.]

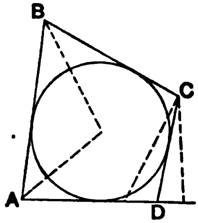


46. Of all triangles having a common base and an equal altitude, the isosceles triangle has the greatest vertical angle.

47. Given the base, the vertical angle, and the foot of the altitude, construct the triangle.

48. If the sum of one pair of opposite sides of a quadrilateral is equal to the sum of the other pair, a circle can be inscribed in the quadrilateral.

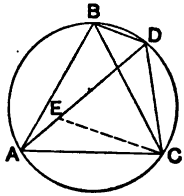
[Describe a ⊙ tangent to three of the sides. Show, by § 396, that the fourth side can neither cut this circle nor lie without it.]



49. Any point on the circumference circumscribing an equilateral triangle is joined with the three vertices.

Prove that the greatest of the three lines is equal to the sum of the other two.

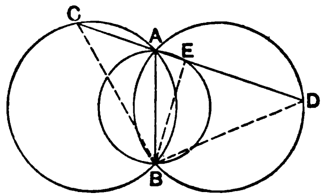
[Lay off $DE = DC$. Prove ⊿ AEC and BDC equal in all respects.]



50. Two equal circles intersect at A and B. On the common chord AB as a diameter a third circle is described. Through A any line CD is drawn terminating in the circumferences and intersecting the third circumference at E.

Prove that CD is bisected at E.

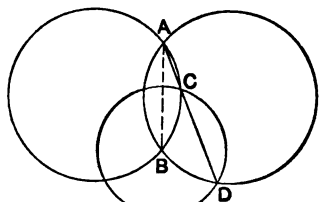
[Show that △ BCD is isosceles, and that BE is ⊥ to the base CD.]



51. Two equal circles intersect at A and B. With B as a center, any circle is described cutting the two equal circumferences at C and D.

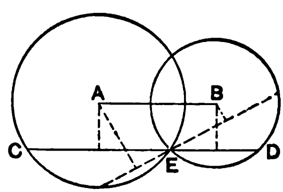
Prove that A, C, and D are in a straight line.

[Draw AC. $\angle BAC \sim \frac{1}{2} BC$. But $BC = BD$. Draw AD. $\angle BAD \sim \frac{1}{2} BD$. $\therefore \angle BAC = \angle BAD$.]



52. If two circles intersect, the longest common secant that can be drawn through either point of intersection is the one that is parallel to the line joining their centers.

[Show that $CD = 2\,AB$, and that any other secant through E is less than $2\,AB$.]



BOOK III

404. Definitions. A *proportion* is the equality of ratios. $\frac{a}{b}=\frac{c}{d}$ is a proportion, and expresses the fact that the ratio of a to b is equal to the ratio of c to d. The proportion $\frac{a}{b}=\frac{c}{d}$ may also be written $a:b=c:d$ and $a:b::c:d$.

In the proportion $\frac{a}{b}=\frac{c}{d}$, the first and fourth terms (a and d) are called the *extremes*, and the second and third terms (b and c) are called the *means*. The first and third terms (a and c) are the *antecedents*, and the second and fourth terms (b and d) are the *consequents*.

In the proportion $\frac{a}{b}=\frac{c}{d}$, d is called a *fourth proportional* to the three quantities a, b, and c.

If the means of a proportion are equal, either mean is a *mean proportional* or a *geometrical mean* between the extremes. Thus in the proportion $\frac{a}{b}=\frac{b}{c}$, b is a mean proportional between a and c. In this same proportion, c is called a *third proportional* to a and b.

Proposition I. Theorem

405. *In a proportion, the product of the extremes is equal to the product of the means.*

Let $$\frac{a}{b}=\frac{c}{d}. \quad (1)$$

To Prove $$ad=bc.$$

Proof. [Clear fractions in (1) by multiplying both members by bd.] Q.E.D.

406. Corollary. *The mean proportional between two quantities is equal to the square root of their product.*

Let $\frac{a}{b}=\frac{b}{c}.$ (1)

To Prove $b=\sqrt{ac}.$

Proof. [Apply § 405 to (1), and extract the square root of both members.] Q.E.D.

407. Exercise. Find x in $\frac{7}{12}=\frac{14}{x}.$

408. Exercise. What is the geometrical mean or mean proportional between 9 and 4?

409. Exercise. 12 is the geometrical mean between two numbers. One of them is 16. What is the other?

410. Exercise. Find the mean proportional between $a^2+2ab+b^2$ and $a^2-2ab+b^2$.

Proposition II. Theorem. (Converse of Prop. I.)

411. *If the product of two factors is equal to the product of two other factors, the factors of either product may be made the means, and the factors of the other product the extremes of a proportion.*

Let $ad=bc.$ (1)

To Prove $\frac{a}{b}=\frac{c}{d}.$

Proof. [Divide both members of (1) by bd.] Q.E.D.

412. Exercise. From the equation $ad=bc$, derive the following eight proportions.

$$\frac{a}{b}=\frac{c}{d},\quad \frac{a}{c}=\frac{b}{d},\quad \frac{c}{d}=\frac{a}{b},\quad \frac{c}{a}=\frac{d}{b},$$

$$\frac{b}{a}=\frac{d}{c},\quad \frac{b}{d}=\frac{a}{c},\quad \frac{d}{c}=\frac{b}{a},\quad \frac{d}{b}=\frac{c}{a}.$$

413. Exercise. Form different proportions from

$$xy = a^2 - b^2.$$

414. Exercise. Form a proportion from

$$a^2 + 2ab + b^2 = my.$$

What is $a + b$ called in this proportion?

415. Exercise. Form a proportion from $a^3 + b^3 = x^2 - y^2$.

416. Definition. A proportion is arranged by *alternation* when antecedent is compared with antecedent and consequent with consequent.

If the proportion $\frac{a}{b} = \frac{c}{d}$ is arranged by alternation, it becomes $\frac{a}{c} = \frac{b}{d}$.

Proposition III. Theorem

417. *If four quantities are in proportion, they are in proportion by alternation.*

Let $$\frac{a}{b} = \frac{c}{d}. \quad (1)$$

To Prove $$\frac{a}{c} = \frac{b}{d}.$$

Proof. Apply § 405 to (1) $ad = bc$. (2)

Apply § 411 to (2) $\frac{a}{c} = \frac{b}{d}$. Q.E.D.

418. Exercise. Write a proportion that will not be altered when arranged by alternation.

419. Definition. A proportion is arranged by *inversion* when the antecedents are made consequents, and the consequents are made antecedents.

If the proportion $\frac{a}{b} = \frac{c}{d}$ is arranged by inversion, it becomes $\frac{b}{a} = \frac{d}{c}$.

Proposition IV. Theorem

420. *If four quantities are in proportion, they are in proportion by inversion.*

Let $\frac{a}{b} = \frac{c}{d}$. (1)

To Prove $\frac{b}{a} = \frac{d}{c}$.

Proof. Apply § 405 to (1) $ad = bc$. (2)

Apply § 411 to (2) $\frac{b}{a} = \frac{d}{c}$. Q.E.D.

421. Definition. A proportion is arranged by *composition* when the sum of antecedent and consequent is compared with either antecedent or consequent.

The proportion $\frac{a}{b} = \frac{c}{d}$ arranged by composition becomes

$$\frac{a+b}{a} = \frac{c+d}{c} \text{ or } \frac{a+b}{b} = \frac{c+d}{d}.$$

Proposition V. Theorem

422. *If four quantities are in proportion, they are in proportion by composition.*

Let $\frac{a}{b} = \frac{c}{d}$. (1)

To Prove $\frac{a+b}{a} = \frac{c+d}{c}$.

Proof. Apply § 405 to (1)

$$ad = bc. \quad (2)$$

Add ac to both members of (2)

$$ac + ad = ac + bc. \quad (3)$$

Factor (3) $a(c+d) = c(a+b)$. (4)

Apply § 411 to (4)

$$\frac{a+b}{a} = \frac{c+d}{c}.$$

Q.E.D.

423. Note. The student may discover for himself the steps of the solution of this and the succeeding propositions by studying the *analysis* of the theorem.

In the *analysis* we assume the conclusion (the part to be proved) to be a true equation. Working upon this conclusion by algebraic transformations, we produce the hypothesis.

The *solution* of the theorem begins with the last step of the analysis and *reverses* the work, step by step, until the first step or conclusion is reached.

In § 422 we have given $\frac{a}{b}=\frac{c}{d}$. (1)

We are to prove $\frac{a+b}{a}=\frac{c+d}{c}$. (2)

Analysis

Clear fractions in (2) $c(a+b)=a(c+d)$. (3)

Expand (3) $ac+bc=ac+ad$. (4)

Subtract ac from both members of (4).

$bc=ad$. (5)

Apply § 411 to (5) $\frac{a}{b}=\frac{c}{d}$. (6)

Let the student show that the solution of Prop. V. as given on the preceding page may be obtained by reversing the steps of this analysis.

424. Exercise. Let $\frac{a}{b}=\frac{c}{d}$.

To Prove $\frac{a+b}{b}=\frac{c+d}{d}$.

425. Exercise. Arrange $\frac{a-b}{b}=\frac{c-d}{d}$ by composition.

426. Exercise. Arrange $\frac{2x-4}{4}=\frac{8-x}{x}$ by composition and then find the value of x.

427. Definition. A proportion is arranged by *division* when the difference between antecedent and consequent is compared with either antecedent or consequent.

The proportion $\frac{a}{b}=\frac{c}{d}$ arranged by division becomes

$\frac{a-b}{a}=\frac{c-d}{c}$ or $\frac{a-b}{b}=\frac{c-d}{d}$ or $\frac{b-a}{a}=\frac{d-c}{c}$ or $\frac{b-a}{b}=\frac{d-c}{d}$.

Proposition VI. Theorem

428. *If four quantities are in proportion, they are in proportion by division.*

Let $$\frac{a}{b}=\frac{c}{d}. \qquad (1)$$

To Prove $$\frac{a-b}{a}=\frac{c-d}{c}. \qquad (2)$$

Proof. [*Analysis.* Clear fractions in (2)

$$c(a-b)=a(c-d). \qquad (3)$$

Expand (3) $$ac-bc=ac-ad. \qquad (4)$$

Subtract ac from both members of (4)

$$-bc=-ad. \qquad (5)$$

Divide both members of (5) by -1

$$bc=ad. \qquad (6)$$

Apply § 411 to (6) $$\left.\frac{a}{b}=\frac{c}{d}\right].$$ Q.E.D.

Let the student derive the *solution* of Prop. VI. from the analysis.

429. Exercise. If $$\frac{a+b-c}{c+d+a}=\frac{a-c}{2d},$$

then $$\frac{b}{a-c}=\frac{a+c-d}{2d}.$$

430. Definition. A proportion is arranged by *composition and division,* when the sum of antecedent and consequent is compared with the difference of antecedent and consequent.

The proportion $\frac{a}{b}=\frac{c}{d}$, arranged by composition and division, becomes

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

PROPOSITION VII. THEOREM

431. *If four quantities are in proportion, they are in proportion by composition and division.*

Let $$\frac{a}{b}=\frac{c}{d}.$$

To Prove $$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

Proof. [Analyze and solve.]

432. EXERCISE. If $$\frac{a}{b}=\frac{c}{d},$$

prove $$\frac{c-a}{a+c}=\frac{d-b}{b+d}.$$

PROPOSITION VIII. THEOREM

433. *If four quantities are in proportion, like powers of those quantities are proportional.*

Let $$\frac{a}{b}=\frac{c}{d}. \qquad (1)$$

To Prove $$\frac{a^n}{b^n}=\frac{c^n}{d^n}.$$

Proof. [Raise both members of (1) to the nth power.] Q.E.D.

434. COROLLARY. *If four quantities are in proportion, like roots of those quantities are proportional.*

435. EXERCISE. If $$\frac{a}{b}=\frac{c}{d},$$

show that $$\frac{a^2}{c^2}=\frac{a^2-b^2}{c^2-d^2}.$$

436. EXERCISE. If $$\frac{a}{b}=\frac{c}{d},$$

show that $$\frac{a^3+b^3}{a^3-b^3}=\frac{c^3+d^3}{c^3-d^3}.$$

Proposition IX. Theorem

437. *If four quantities are in proportion, equimultiples of the antecedents are proportional to equimultiples of the consequents.*

Let $$\frac{a}{b}=\frac{c}{d}. \quad (1)$$

To Prove $$\frac{ax}{by}=\frac{cx}{dy}.$$

Proof. Multiply both members of (1) by $\frac{x}{y}$. Q.E.D.

438. Exercise. **Let** $$\frac{a}{b}=\frac{c}{d}.$$

To Prove $$\frac{ac}{bd}=\frac{c^2}{d^2}.$$

439. Exercise. **Let** $$\frac{a}{b}=\frac{c}{d}.$$

To Prove $$\frac{ab+cd}{ab-cd}=\frac{a^2+c^2}{a^2-c^2}.$$

440. Exercise. **Let** $$\frac{a}{b}=\frac{b}{c}.$$

To Prove $$\frac{a+c}{a-c}=\frac{b^2+c^2}{b^2-c^2}.$$

441. Exercise. **Let** $$\frac{a}{b}=\frac{c}{d}.$$

To Prove $$\frac{ma^2+nc^2}{mb^2+nd^2}=\frac{a^2}{b^2}.$$

442. Definition. A continued proportion is a proportion made up of several ratios that are successively equal to each other. Example:

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}, \text{ etc.}$$

Proposition X. Theorem

443. *In a continued proportion the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Let $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}. \qquad (1)$$

To Prove $$\frac{a+c+e+g}{b+d+f+h}=\frac{e}{f}.$$

Proof
$$\left.\begin{aligned}\frac{a}{b}&=\frac{e}{f} \quad (2)\\ \frac{c}{d}&=\frac{e}{f} \quad (3)\\ \frac{e}{f}&=\frac{e}{f} \quad (4)\\ \frac{g}{h}&=\frac{e}{f} \quad (5)\end{aligned}\right\}\text{From (1).}$$

$$\left.\begin{aligned}af&=be \quad (6)\\ cf&=de \quad (7)\\ ef&=fe \quad (8)\\ gf&=he \quad (9)\end{aligned}\right\}\text{From (2), (3), (4), and (5).}$$

Add (6), (7), (8), and (9), and factor.

$$f(a+c+e+g)=e(b+d+f+h). \qquad (10)$$

Apply § 411 to (10).

$$\frac{a+c+e+g}{b+d+f+h}=\frac{e}{f}.$$ Q.E.D.

444. Exercise. If $$\frac{x}{a}=\frac{y}{b}=\frac{z}{c},$$

then will $$\frac{x+y}{a+b}=\frac{y+z}{b+c}=\frac{z+x}{c+a}.$$

445. Exercise. **Let** $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}.$$

To Prove $$\frac{a-c+e-g}{b-d+f-h}=\frac{c}{d}.$$

Proposition XI. Theorem

446. *If the terms of one proportion are multiplied by the corresponding terms of another proportion, the products are proportional.*

Let $\frac{a}{b}=\frac{c}{d}$ (1) and $\frac{x}{y}=\frac{m}{n}$ (2).

To Prove $\frac{ax}{by}=\frac{cm}{dn}$.

Proof. [The proof is left to the student.] Q.E.D.

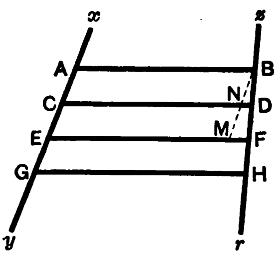
447. Exercise. If the terms of one proportion are divided by the corresponding terms of another proportion, the quotients are proportional.

448. Exercise. If $\frac{a}{b}=\frac{c}{d}$,

show that $\frac{a^2+ab+b^2}{a^2-ab+b^2}=\frac{c^2+cd+d^2}{c^2-cd+d^2}$.

Proposition XII. Theorem

449. *If a number of parallels intercept equal distances on one of two transversals, they will intercept equal distances on the other also.*

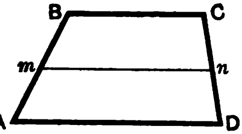


Let AB, CD, EF, and GH be a number of parallels cut by the transversals xy and zr, making

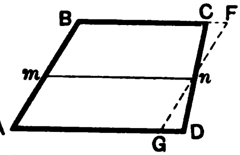
$$AC = CE = EG.$$

To Prove $BD = DF = FH$.

Proof. [Proof similar to that of § 240.] Q.E.D.



450. Corollary I. *A line drawn from the middle point of one of the inclined sides of a trapezoid parallel to either base, bisects the other inclined side.*



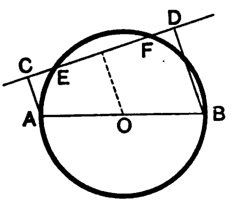
451. Corollary II. *A line joining the middle points of the inclined sides of a trapezoid is parallel to the bases.*

Suggestion. Draw $FG \parallel AB$. Prove $\triangle CFn = \triangle GDn$ whence $Fn = nG$. Prove $FG = AB$ and $nG = Am$. Prove $AmnG$ a parallelogram.

452. Exercise. A line joining the middle points of two opposite sides of a parallelogram, is parallel to the two remaining sides and passes through the point of intersection of the diagonals.

453. Exercise. A line joining the middle points of the inclined sides of a trapezoid is equal to one half the sum of the parallel sides.

[In the figure of § 451 show $mn = \frac{1}{2}(BF + AG)$ and $CF = GD$].



454. Exercise. If from the extremities of a diameter perpendiculars are drawn to a line cutting the circle, the parts intercepted between the feet of the perpendiculars and the curve are equal.

[To prove $CE = FD$.]

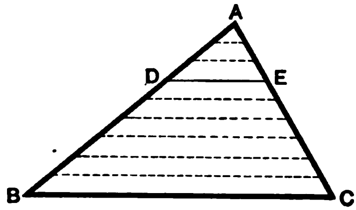
455. Exercise. If perpendiculars are drawn from the extremities of a diameter of a circle to a line lying without the circle, the feet of these perpendiculars are equally distant from the center of the circle.

456. Exercise. A line joining the middle points of the inclined sides of a trapezoid bisects the diagonals of the trapezoid, and also bisects any line whose extremities are in the parallel bases.

457. Exercise. The inclined sides of a trapezoid are 9 ft. and 15 ft. respectively. If on the shorter of these sides a point is taken 3 ft. from one end, and through that point a parallel to either base is drawn, where does the parallel intersect the other inclined side?

Proposition XIII. Theorem.

458. ***A line drawn parallel to one side of a triangle divides the other two sides proportionally.***



Let DE be parallel to BC.

To Prove $$\frac{AD}{DB}=\frac{AE}{EC}.$$

Proof. Case I. When the segments AD and DB are commensurable.

Let the common unit of measure be contained in AD m times, and in DB n times.

Whence $$\frac{AD}{DB}=\frac{m}{n}. \quad (1)$$

Divide AD into m equal parts, each equal to the unit of measure, and DB into n equal parts, and through the points of division draw parallels to BC.

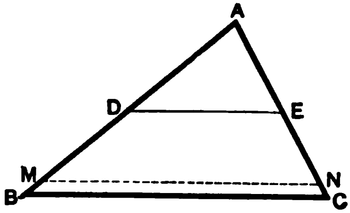
These parallels intercept equal distances on AC (?). Consequently AE is divided into m equal parts, and EC into n equal parts.

Whence $$\frac{AE}{EC}=\frac{m}{n}. \quad (2)$$

Compare (1) and (2).

$$\frac{AD}{DB}=\frac{AE}{EC}.$$ Q.E.D.

CASE II. When the segments AD and DB are incommensurable.



Let DE be parallel to BC.

To Prove $$\frac{AD}{DB} = \frac{AE}{EC}.$$

Proof. Divide AD into a number of equal parts, and let one of these parts be applied to DB as a unit of measure.

Since AD and DB are incommensurable, this unit of measure will not be exactly contained in DB, but there will remain over some distance MB smaller than the unit of measure.

Draw MN parallel to BC.

Since AD and DM are commensurable (why ?),

$$\frac{AD}{DM} = \frac{AE}{EN} \text{ by Case I.}$$

This proportion is true, no matter how many equal divisions are made in AD.

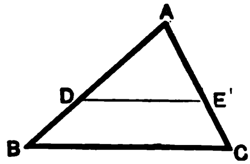
If the number of divisions is increased, the size of each division is diminished, and MB is also diminished.

As the number of divisions is increased, the ratio $\frac{AD}{DM}$ is approaching $\frac{AD}{DB}$ as its limit, and the ratio $\frac{AE}{EN}$ is approaching $\frac{AE}{EC}$ as its limit.

Since the variables $\frac{AD}{DM}$ and $\frac{AE}{EN}$ are always equal, and are each approaching a limit, their limits are equal (?).

Therefore $$\frac{AD}{DB} = \frac{AE}{EC}.$$ Q.E.D.

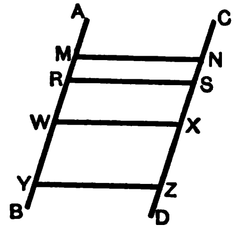
459. Corollary I. *DE is parallel to BC.*



To Prove $\frac{AD}{AB}=\frac{AE}{AC}$ and $\frac{DB}{AB}=\frac{EC}{AC}$.

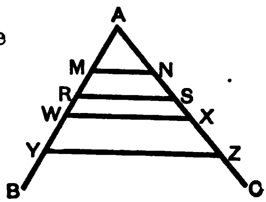
Suggestion. Apply § 422 to $\frac{AD}{DB}=\frac{AE}{EC}$.

460. Corollary II. *If two lines are cut by any number of parallels, they are divided proportionally.*



Case I. When the two lines are parallel.

To Prove $\frac{MR}{NS}=\frac{RW}{SX}=\frac{WY}{XZ}$.

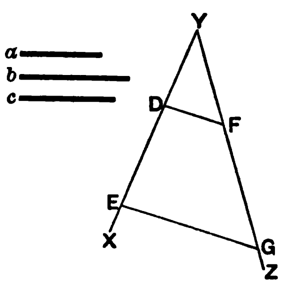


Case II. When the two lines are oblique to each other.

To Prove $\frac{AM}{AN}=\frac{MR}{NS}=\frac{RW}{SX}=\frac{WY}{XZ}$.

Use § 458 and § 459.

461. Corollary III. *To construct a fourth proportional to three given lines.*



Let *a*, *b*, and *c* be the three given lines.

Required to construct a fourth proportional to them.

Construct any convenient angle, *XYZ*.

Lay off $YD=a$, $DE=b$, and $YF=c$.

Draw *DF*. Draw *EG* ∥ to *DF*.

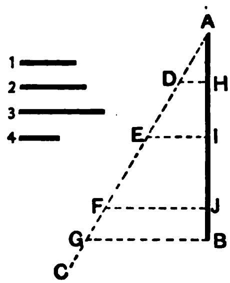
FG is the required fourth proportional.

$$\frac{YD}{DE} = \frac{YF}{FG} \text{ (?) or } \frac{a}{b} = \frac{c}{FG}. \qquad \text{Q.E.F.}$$

Note. If *b* and *c* are equal, *FG* is a *third proportional* to *a* and *b*.

462. Corollary IV. *To divide a line into parts proportional to given lines.*

Let *AB* be the given line.



Required to divide it into parts proportional to the lines 1, 2, 3, and 4.

Draw *AC*, making any convenient angle with *AB*. Lay off $AD = 1$, $DE = 2$, $EF = 3$, and $FG = 4$.

Connect *G* and *B*.

Through *F*, *E*, and *D* draw parallels to *GB*.

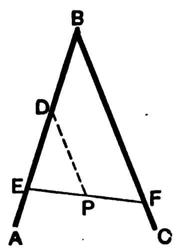
Then $$\frac{AH}{AD} = \frac{HI}{DE} = \frac{IJ}{EF} = \frac{JB}{FG},$$

or $$\frac{AH}{1} = \frac{HI}{2} = \frac{IJ}{3} = \frac{JB}{4}. \qquad \text{Q.E.F.}$$

463. Exercise. In the triangle *ABC*, *AB* is 10 in. and *AC* is 8 in. From a point *D* on the line *AB*, *DE* is drawn parallel to *BC*, making $AD = 3$ in. Find the lengths of *AE* and *EC*.

464. Exercise. Through the point of intersection of the medians of a triangle, a line is drawn parallel to any side of the triangle. How does it divide each of the other two sides of the triangle?

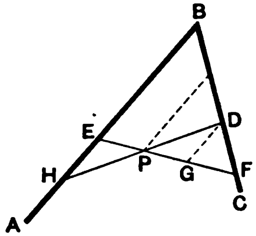
Suggestion. Use § 245.



465. Exercise. Through a point within an angle draw a line limited by the sides of the angle and bisected by the point.

Through the given point, *P*, draw *PD* ∥ to *BC*, and lay off $DE = DB$.

466. Exercise. ABC is any angle and P a point within. To draw through P a line limited by the sides of the angle, and cutting off a triangle whose area is a minimum.



Draw HD so that $HP = PD$.

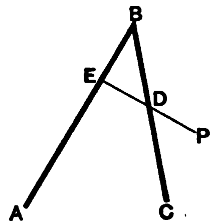
$\triangle HBD$ is the minimum $\triangle$.

Draw any other line through P, as EF.

Draw $DG \parallel$ to BA.

$\triangle PEH = \triangle PGD$. $\therefore \triangle EBF$ exceeds area of $\triangle HBD$ by $\triangle DGF$.

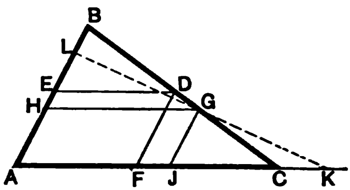
467. Exercise. Construct a fourth proportional to three lines in the ratio of 2, 3, and 4.



468. Exercise. Construct a third proportional to two lines whose lengths are 1 in. and 3 in. respectively.

469. Exercise. Through a point P without an angle ABC, draw PE so that $PD = DE$.

470. Exercise. In the triangle ABC, D is the middle point of BC and G is any other point on BC. Prove that the parallelogram $DEAF$ is greater than the parallelogram $GHAJ$.



Suggestion. Draw LK so that $LG = GK$.

$$\triangle ABC > \triangle ALK, \quad (?) \qquad DEAF = \tfrac{1}{2} \triangle ABC, \quad (?)$$

and

$$GHAJ = \tfrac{1}{2} \triangle ALK. \quad (?)$$

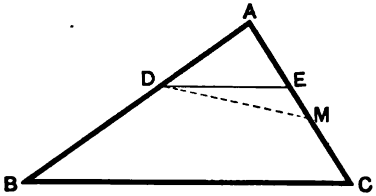
471. Exercise. Divide a line into any number of equal parts, using the principle of this proposition. Compare the method with that used in § 240.

472. Exercise. Prove § 239, using the principle established in this proposition.

473. Exercise. If an equilateral triangle is inscribed in a circle, and through the center of the circle lines are drawn parallel to the sides of the triangle, these lines trisect the sides of the triangle.

PROPOSITION XIV. THEOREM (CONVERSE OF PROP. XIII.)

474. *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*



Let $\frac{AD}{DB}=\frac{AE}{EC}$.

To Prove DE parallel to BC.

Proof. Suppose DE is not parallel to BC and that any other line through D, as DM, is parallel to BC.

$$\frac{AD}{DB}=\frac{AM}{MC}. \quad (?)$$

$$\frac{AD}{DB}=\frac{AE}{EC}. \quad (?)$$

$$\frac{AM}{MC}=\frac{AE}{EC}. \quad (?)$$

Show that this last proportion is absurd.

Therefore the supposition that DE is not parallel to BC is false. Q.E.D.

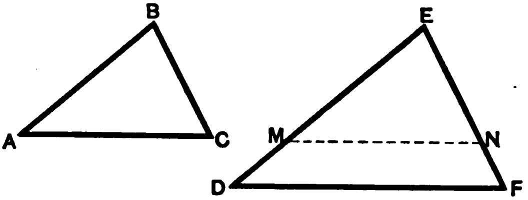
475. COROLLARY. *If* $\frac{AD}{AB}=\frac{AE}{AC}$, *$DE$ and BC are parallel.*

476. EXERCISE. *DE* is drawn, cutting the sides *AB* and *AC* of a triangle *ABC* at *D* and *E*. The segment *BD* is ¼ of *AB*, and *AE* is ¾ of *AC*. Show that *DE* and *BC* are parallel.

477. DEFINITION. Two polygons are similar when they are mutually equiangular, and have their sides about the equal angles tăken in the same order proportional.

PROPOSITION XV. THEOREM

478. ***Triangles that are mutually equiangular are similar.***



Let ABC and DEF be two $\triangle$ having $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.

To Prove $\triangle ABC$ and DEF similar.

Proof. Lay off $EM = BA$, $EN = BC$. Draw MN.

Prove $\triangle ABC$ and MEN equal in all respects.

Whence $\angle M = \angle D$.

MN and DF are $\parallel$. (?)

$$\frac{EM}{ED} = \frac{EN}{EF} \quad (?) \quad \text{or} \quad \frac{AB}{DE} = \frac{BC}{EF}.$$

In a similar manner prove $\frac{AB}{DE} = \frac{AC}{DF}$, and $\frac{BC}{EF} = \frac{AC}{DF}$.

The triangles are by hypothesis mutually equiangular, and we have proved their sides proportional, therefore by definition they are similar. Q.E.D.

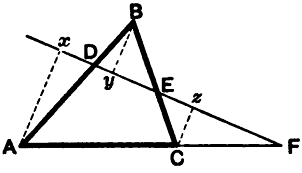
479. COROLLARY. *Two triangles are similar if they have two angles of one equal respectively to two angles of the other.*

480. EXERCISE. All equilateral triangles are similar.

481. EXERCISE. Are all isosceles triangles similar? Are right-angled isosceles triangles similar?

482. Exercise. If the sides of a triangle ABC be cut by any transversal, in the points D, E, and F, to prove

$$\frac{AD}{DB}\times\frac{BE}{EC}\times\frac{CF}{FA}=1.$$



[From A, B, and C, draw perpendiculars to the transversal. Show that ▵ AxD and DyB are similar,

whence $$\frac{AD}{DB}=\frac{Ax}{By}. \quad (1)$$

Similarly, $$\frac{BE}{EC}=\frac{By}{Cz}, \quad (2)$$

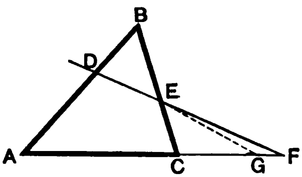
and $$\frac{CF}{FA}=\frac{Cz}{Ax}. \quad (3)$$

Multiply (1), (2), and (3) together, member by member.]

Note. Prove this exercise when the points D, E, and F are all external, *i.e.* are all on the prolonged sides of the triangle. (If the figure be lettered as above, the proportions in the proof of this case will be precisely like the foregoing.)

483. Exercise. If D, E and F are three points on the sides of a triangle, either all external, or two internal and one external, such that

$$\frac{AD}{DB}\times\frac{BE}{EC}\times\frac{CF}{FA}=1,$$



the three points are in the same line.

[Draw DE and EF. Let any other line than EF as EG be the prolongation of DE. By the preceding exercise

$$\frac{AD}{DB}\times\frac{BE}{EC}\times\frac{CG}{GA}=1. \quad (1)$$

By hypothesis $$\frac{AD}{DB}\times\frac{BE}{EC}\times\frac{CF}{FA}=1. \quad (2)$$

From (1) and (2) we derive

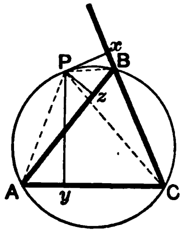
$$\frac{CG}{GA}=\frac{FC}{FA}. \quad (3)$$

Arrange (3) by division,

$$\frac{CG}{GA - CG} = \frac{FC}{FA - FC}, \text{ or } \frac{CG}{AC} = \frac{FC}{AC}.$$

Whence $CG = FC$ which is absurd.

∴ the supposition that any other line than EF is the prolongation of DE is absurd.]



484. Exercise. If from any point on the circumference of a circle circumscribed about a triangle perpendiculars be drawn to the three sides of the triangle, the feet of these perpendiculars are in the same straight line.

[To prove x, y, and z are in a straight line.

Connect P with the three vertices.

By means of similar triangles, show:

$$\frac{Az}{Cx} = \frac{Pz}{Px}, \tag{1}$$

$$\frac{Cy}{Bz} = \frac{Py}{Pz}, \tag{2}$$

$$\frac{Bx}{Ay} = \frac{Px}{Py}. \tag{3}$$

Multiply (1), (2), and (3) together, member by member,

$$\frac{Az}{Cx} \times \frac{Cy}{Bz} \times \frac{Bx}{Ay} = 1,$$

or

$$\frac{Az}{zB} \times \frac{Bx}{xC} \times \frac{Cy}{yA} = 1.$$

By the preceding exercise, x, y, and z are in the same straight line.]

485. Exercise. If a triangle ABC be inscribed in a circle, tangents to this circle at A, B, and C meet BC, CA, and AB respectively in three points that are in the same straight line.

[Let the tangents meet BC, CA, and AB in the points x, y, and z respectively. Prove ▲ AzC and BzC similar.

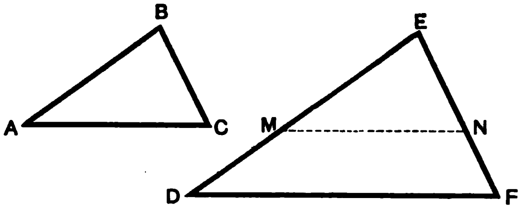
Whence $\quad \frac{Az}{AC} = \frac{zC}{BC}, \; (1)$ and $\frac{BC}{Bz} = \frac{AC}{Cz}. \; (2)$

Combining (1) and (2), $\quad \frac{Az}{zB} = \frac{\overline{AC}^2}{\overline{BC}^2}.$

Similarly, $\quad \frac{Bx}{xC} = \frac{\overline{AB}^2}{\overline{AC}^2}$, and $\frac{Cy}{yA} = \frac{\overline{BC}^2}{\overline{AB}^2}.$]

PROPOSITION XVI. THEOREM

486. *Triangles that have their corresponding sides proportional are similar.*



Let ABC and DEF be two △ having

$$\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}.$$

To Prove △ ABC and DEF similar.

Proof. Lay off $EM = BA$ and $EN = BC$. Draw MN.

Show that $$\frac{EM}{ED}=\frac{EN}{EF}.$$

MN is parallel to DF. (?)
Prove △ EMN and EDF similar.

Whence $$\frac{EN}{EF}=\frac{MN}{DF}. \qquad (1)$$

By hypothesis $$\frac{BC}{EF}=\frac{AC}{DF}. \qquad (2)$$

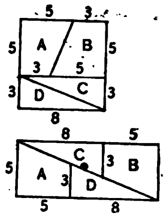
Compare (1) and (2), remembering that $BC=EN$, and show that $AC = MN$.

Prove △ ABC and MEN equal in all respects.

△ DEF and MEN have been proved similar, and since △ ABC and MEN are equal in all respects, △ DEF and ABC are similar. Q.E.D

487. Exercise. The sides of a triangle are 6 in., 8 in., and 12 in. respectively. The sides of a second triangle are 6 in., 3 in., and 4 in. respectively. Are they similar?

488. Scholium. Polygons must fulfill two conditions in order to be similar, *i.e.* they must be mutually equiangular, and must have their corresponding sides proportional. Propositions XV. and XVI. show that in the case of triangles, either of these conditions involves the other. Hence to prove *triangles* similar, it will be sufficient to show either that they are mutually equiangular, or that their corresponding sides are proportional.



489. Exercise. A piece of cardboard 8 in. square is cut into 4 pieces, *A*, *B*, *C*, and *D*, as shown in the first figure. These pieces, as placed in the second figure, *apparently*, form a rectangle whose area is 65 sq. in.

Explain the fallacy by means of similar triangles.

490. Exercise. The sides of a triangle are 12, 16, and 24 ft. respectively. A similar triangle has one side 8 ft. in length. What is the length of the other two sides? (Three solutions.)

491. Exercise. On a given line as a side construct a triangle similar to a given triangle. [Construct in two ways. Use § 478 and also § 486.]

492. Exercise. Construct a triangle that shall have a given perimeter, and shall be similar to a given triangle.

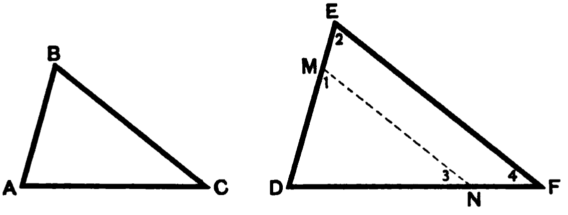
493. Exercise. If the sides of one triangle are *inversely proportional* to the sides of a second triangle, the triangles are not necessarily similar.

[Let the sides of the first triangle be in the ratio of 2, 3, and 4. Then the sides of the second triangle are in the ratio of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, or $\frac{6}{12}$, $\frac{4}{12}$, and $\frac{3}{12}$; and these fractions are in the ratio of the integers 6, 4, and 3. Therefore the triangles are not similar.]

494. Exercise. Any two altitudes of a triangle are inversely proportional to the sides to which they are respectively perpendicular.

Proposition XVII. Theorem.

495. ***Triangles that have an angle in each equal, and the including sides proportional, are similar.***



Let ⊿s ABC and DEF have $\angle A = \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$.

To Prove ⊿s ABC and DEF similar.

Proof. Lay off $DM = AB$ and $DN = AC$. Draw MN.

Prove ⊿s ABC and DMN equal in all respects.

$$\frac{DM}{DE} = \frac{DN}{DF}. \quad (?)$$

MN and EF are parallel. (?)

$\angle 1 = \angle 2$ and $\angle 3 = \angle 4$. (?)

⊿s DMN and DEF are similar. (?)

⊿s ABC and DEF are similar. (?) Q.E.D.

496. Exercise. If a line is drawn parallel to the base of a triangle, and lines are drawn from the vertex to different points of the base, these lines divide the base and the parallel proportionally.

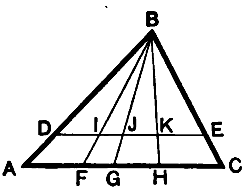
⊿s DBI and ABF are similar. (?)

$$\therefore \frac{DI}{AF} = \frac{BI}{BF}.$$

⊿s IBJ and FBG are similar.

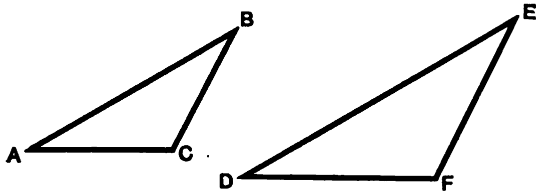
$$\therefore \frac{IJ}{FG} = \frac{BI}{BF}.$$

$$\therefore \frac{DI}{AF} = \frac{IJ}{FG}, \text{ etc.}$$



Proposition XVIII. Theorem.

497. *Triangles that have their sides parallel, each to each, or perpendicular, each to each, are similar.*



Let △s ABC and DEF have $AB \parallel$ to DE, $BC \parallel$ to EF, and $AC \parallel$ to DF.

To Prove △s ABC and DEF similar.

Proof. The angles of the △ ABC are either equal to the angles of △ DEF, or are their supplements. (§ 131 and § 132.)

There are four possible cases:

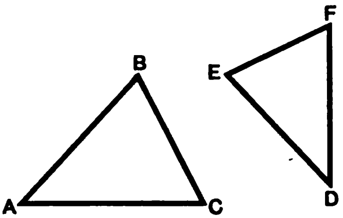
1. The three angles of △ ABC may be supplements of the angles of △ DEF.
2. Two angles of △ ABC may be supplements of two angles of △ DEF, and the third angle of △ ABC equal the third angle of △ DEF.
3. One angle of △ ABC may be the supplement of an angle of △ DEF, and the two remaining angles of △ ABC be equal to the two remaining angles of △ DEF.
4. The three angles of △ ABC may equal the three angles of △ DEF.

Show that in the first case the sum of the angles of the two △s would be six right angles.

Show that in the second case the sum of the angles of the two △s would be greater than four right angles.

Show, by means of § 140, that the third case is impossible unless the angles that are supplementary are right angles, in which case they would also be equal, and the triangles would have three angles of the one equal to three angles of the other.

Therefore if two triangles have their sides parallel, each to each, the triangles are mutually equiangular, and consequently similar.



Let △ ABC and DEF have $AB \perp DE$, $BC \perp EF$, and $AC \perp DF$.

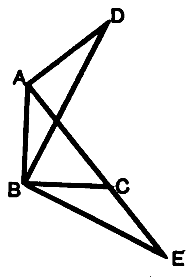
To Prove △ ABC and DEF similar.

Proof. The angles of △ ABC are either equal to the angles of △ DEF, or are their supplements.

[Show, as was done in the first part of this proposition, that the angles of △ ABC are equal to those of △ DEF, and consequently △ ABC and DEF are similar.] Q.E.D.

Note. The equal angles are those that are included between sides that are respectively parallel or perpendicular to each other.

498. Exercise. The bases of a trapezoid are 8 in. and 12 in., and the altitude is 6 in. Find the altitudes of the two triangles formed by producing the non-parallel sides until they meet.



499. Exercise. The angles ABC, DAE, and DBE are right angles.

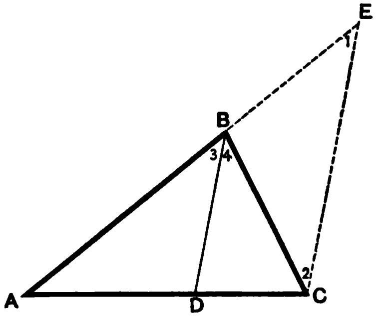
Prove that two triangles in the diagram are similar.

500. Exercise. The lines joining the middle points of the sides of a given triangle form a second triangle that is similar to the given triangle.

501. Exercise. The bisectors of the exterior angles of an equilateral triangle form by their intersection a triangle that is also equilateral.

Proposition XIX. Theorem.

502. *The bisector of an angle of a triangle divides the opposite side into segments that are proportional to the adjacent sides of the angle.*



Let BD be the bisector of $\angle B$ of the $\triangle ABC$.

To Prove $\frac{AD}{DC} = \frac{AB}{BC}$.

Proof. Prolong AB until $BE = BC$. Draw CE.

$$\angle 3 + \angle 4 = \angle 1 + \angle 2. \quad (?)$$

$$\angle 3 = \angle 4 \text{ and } \angle 1 = \angle 2. \quad (?)$$

$$\angle 4 = \angle 2. \quad (?)$$

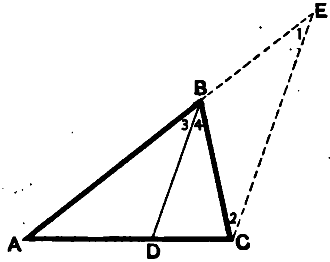
BD and EC are parallel. (?)

$$\frac{AB}{BE} = \frac{AD}{DC}. \quad (?)$$

$$\frac{AB}{BC} = \frac{AD}{DC}. \quad (?)$$

Q.E.D.

Conversely. A line drawn through the vertex of an angle of a triangle, dividing the opposite side into segments proportional to the adjacent sides of the angle, bisects the angle.



Let ABC be a $\triangle$ in which BD is drawn making $\frac{AD}{DC} = \frac{AB}{BC}$.

To Prove that BD bisects $\angle B$.

Proof. Prolong AB until $BE = BC$. Draw EC.

$$\frac{AD}{DC} = \frac{AB}{BC}. \quad (?)$$

$$\frac{AD}{DC} = \frac{AB}{BE}. \quad (?)$$

BD is parallel to EC. (?)

$\angle 3 = \angle 1$, and $\angle 4 = \angle 2$. (?)

Since $\angle 1 = \angle 2$. (?)

$\angle 3 = \angle 4$. (?) Q.E.D.

503. Exercise. The triangle ABC has $AB = 8$ in., $BC = 6$ in., and $AC = 12$ in. BD bisects $\angle B$. What are the lengths of the segments into which it divides AC?

504. Exercise. BD is the bisector of $\angle B$ in the triangle ABC. The segments of AC are $AD = 5$ in. and $DC = 2$ in. The sum of the sides AB and BC is 14 in. Find the lengths of AB and BC.

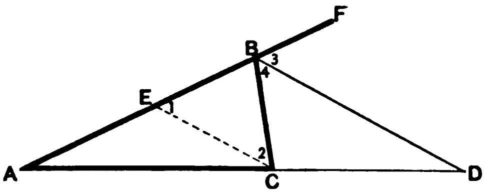
505. Exercise. Construct a triangle having given two sides and one of the two segments into which the third side is divided by the bisector of the opposite angle. (Two constructions.)

506. Definition. A point C, taken on the line AB between the points A and B, is said to divide the line AB *internally* into two segments, CA and CB.

A point C', taken on AB produced, is said to divide AB *externally* into two segments, $C'A$ and $C'B$. In each case, the segments are the distances from C (or C') to the extremities of AB.

PROPOSITION XX. THEOREM

507. ***The bisector of an exterior angle of a triangle divides the opposite side externally into two segments that are proportional to the adjacent sides of the angle.***



Let *BD* bisect the exterior $\angle CBF$ of the $\triangle ABC$.

To Prove $\dfrac{AD}{DC} = \dfrac{AB}{BC}$.

Proof. Lay off $BE = BC$. Draw *EC*.

$$\angle 3 + \angle 4 = \angle 1 + \angle 2. \quad (?)$$

$$\angle 3 = \angle 4, \text{ and } \angle 1 = \angle 2. \quad (?)$$

$$\angle 4 = \angle 2. \quad (?)$$

EC and *BD* are parallel. (?)

$$\frac{AB}{BE} = \frac{AD}{DC}. \quad (?)$$

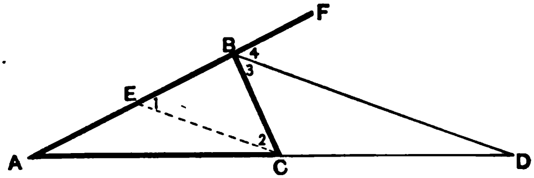
$$\frac{AB}{BC} = \frac{AD}{DC}. \quad (?)$$

Q.E.D.

CONVERSELY. A line drawn through the vertex of an angle of a triangle dividing the opposite side externally into segments proportional to the adjacent sides of the angle, bisects the exterior angle.

Let BD be drawn so that $\frac{AD}{DC} = \frac{AB}{BC}$.

To Prove that BD bisects $\angle CBF$.



Proof. Lay off $BE = BC$. Draw CE.

$$\frac{AD}{DC} = \frac{AB}{BC}. \quad (?) \qquad \frac{AD}{DC} = \frac{AB}{BE}. \quad (?)$$

$$EC \text{ is parallel to } BD. \quad (?)$$

$$\angle 4 = \angle 1, \text{ and } \angle 3 = \angle 2. \quad (?)$$

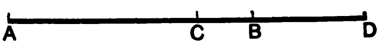
$$\angle 1 = \angle 2. \quad (?)$$

$$\angle 3 = \angle 4. \quad (?)$$

Q.E.D.

508. Exercise. The lengths of the sides of a triangle are 4, 5, and 6 yards, respectively. Find the lengths of the segments into which the bisector of the angle exterior to the largest angle of the triangle divides the opposite side externally.

509. Definition. A line is divided *harmonically* when it is divided internally and externally in the same ratio.



If, in this figure,

$$\frac{AC}{CB} = \frac{AD}{DB},$$

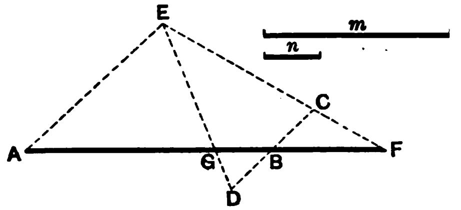
then AB is divided harmonically.

510. Exercise. The bisector of an angle of a triangle and the bisector of its adjacent exterior angle divide the opposite side harmonically. (§§ 502, 507.)

511. Exercise. To divide a line internally and externally so that its segments shall have a given ratio, *i.e.* to divide a line harmonically.

Let AB be the given line, and m and n lines in the given ratio.

Required to divide AB internally and externally into segments having the ratio $\frac{m}{n}$.



Draw AE making any angle with AB, and equal to m.

Draw BC parallel to AE, and equal to n.

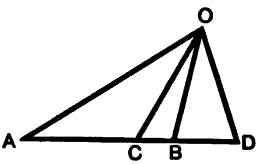
Prolong CB until $BD = n$. Draw ED.

Draw EC and prolong it until it meets AB prolonged at some point F.

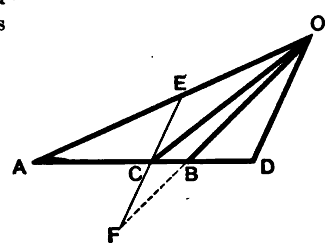
By means of similar triangles, show

$$\frac{AG}{GB} = \frac{m}{n}, \text{ and } \frac{AF}{BF} = \frac{m}{n}; \text{ whence } \frac{AG}{GB} = \frac{AF}{BF}. \qquad \text{Q.E.F.}$$

512. Definition. If the line AB is divided harmonically at C and D, and the four points A, B, C, and D are connected with any other point O, the resulting figure is called a *harmonic pencil*. The point O is called the *vertex* of the pencil, and the four lines OA, OC, OB, and OD are called *rays*.



513. Exercise. O–$ACBD$ is a harmonic pencil. EF is drawn through C parallel to OD, and limited by OB produced. Prove that EF is bisected at C.



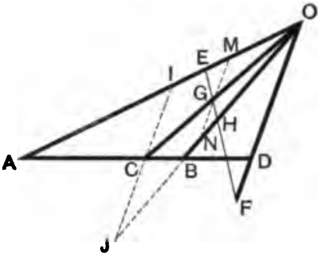
$$\frac{OD}{CF} = \frac{DB}{BC}. \quad (?) \qquad (1)$$

$$\frac{EC}{OD} = \frac{AC}{AD}. \quad (?) \qquad (2)$$

$$\frac{AC}{CB} = \frac{AD}{DB}. \quad (?) \qquad (3)$$

Multiply (1), (2), and (3) together member by member. Q.E.D.

514. Exercise. *O–ACBD* is a harmonic pencil, and *EF* any transversal cutting the rays at *E*, *G*, *H*, and *F*. Prove that the transversal *EH* is divided harmonically, that is,



$$\frac{EG}{GH}=\frac{EF}{FH}.$$

Through *C* draw *IJ* ‖ to *OD*.
Through *G* draw *MN* ‖ to *IJ*.

$$IC = CJ. \quad (?)$$

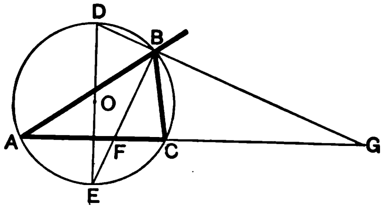
$$\therefore\ MG = GN. \quad (?)$$

$$\frac{EG}{GM}=\frac{EF}{OF}. \quad (?) \qquad \frac{EG}{GN}=\frac{EF}{OF}. \quad (?) \qquad \frac{GH}{GN}=\frac{HF}{OF}. \quad (?)$$

$$\therefore\ \frac{EG}{GH}=\frac{EF}{FH}. \quad (?)$$

Q. E. D.

515. Exercise. *ABC* is an inscribed triangle, *DE* is a diameter perpendicular to *AC*. The vertex *B* is connected with the extremities of the diameter. Prove that *BE* and *DB* (prolonged) divide the base *AC* harmonically.



Suggestion. Show that *BE* and *BG* are the bisectors of ∠ *B* and the exterior angle at *B* respectively.

516. Exercise. Any triangle having *AC* for its base (see figure of § 515), and its other two sides in the ratio $\frac{AB}{BC}$, will have its vertex in the circumference described on *FG* as a diameter.

517. Exercise. The bisectors of the exterior angles of a triangle meet the opposite sides produced in three points that are in the same straight line.

[Let the bisectors of the exterior angles at *A*, *B*, and *C*, of the triangle *ABC*, meet the opposite sides *BC*, *AC*, and *AB* in the points *X*, *Y*, and *Z*, respectively.

$$\frac{AY}{YC}=\frac{AB}{BC}. \quad (?) \qquad \frac{CX}{XB}=\frac{AC}{AB}. \quad (?) \qquad \frac{BZ}{ZA}=\frac{BC}{AC}. \quad (?)$$

Whence

$$\frac{AY}{YC}\times\frac{CX}{XB}\times\frac{BZ}{ZA}=1.]$$

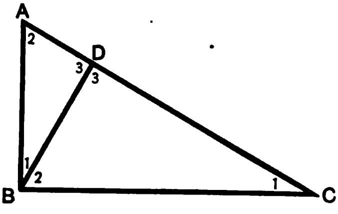
Proposition XXI. Theorem

518. *In a right-angled triangle, if a perpendicular is drawn from the vertex of the right angle to the hypotenuse,*

I. *The triangles on each side of the perpendicular are similar to the original triangle, and to each other.*

II. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

III. *Either side about the perpendicular is a mean proportional between the hypotenuse and the adjacent segment of the hypotenuse.*



Let ABC be a R.A. △, AC its hypotenuse, and $BD \perp$ to AC.

I. **To Prove** ▵ ABD and BDC similar to △ ABC and to each other.

Proof. Show that ▵ ABD and ABC are mutually equiangular, and consequently similar. In the same manner show that ▵ BDC and ABC are similar.

▵ ABD and BDC are also mutually equiangular and similar.

Q.E.D.

II. **To Prove** $\frac{AD}{BD} = \frac{BD}{DC}$.

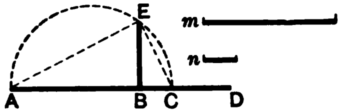
Use the similar ▵ ABD and BDC.

III. **To Prove** $\frac{AC}{AB} = \frac{AB}{AD}$, and $\frac{AC}{BC} = \frac{BC}{DC}$.

Use the similar ▵ ABC and ABD, and also ▵ ABC and BDC.

Q.E.D.

519. COROLLARY. *To construct a mean proportional between two given lines.*



Let m and n be two given lines.

Required to construct a mean proportional between them.

On the indefinite line AD lay off $AB = m$ and $BC = n$.

On AC as a diameter describe a semicircle.

Erect $BE \perp$ to AC.

Draw AE and EC.

Show that AEC is a R.A. △, and that

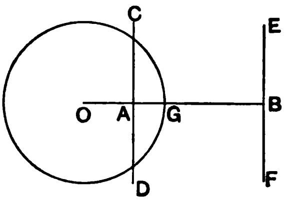
$$\frac{AB}{BE} = \frac{BE}{BC}, \text{ or } \frac{m}{BE} = \frac{BE}{n}.$$

∴ BE is the required mean proportional. Q.E.F.

520. EXERCISE. Construct a third proportional to two given lines by means of Prop. XXI.

521. DEFINITION. If the radius OG is divided internally and externally at A and B, so that

$$OA \times OB = \overline{OG}^2,$$



and through A and B perpendiculars are drawn to OG, each perpendicular is called the *polar* of the other point, which is called in relation to the perpendicular its *pole*.

[EF is the polar of A, and A is the pole of EF.

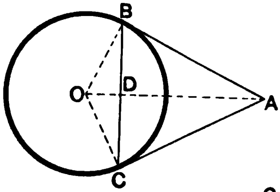
CD is the polar of B, and B is the pole of CD.

Notice that OB is a third proportional to OA and the radius, and OA is a third proportional to OB and the radius.]

522. EXERCISE. Given a point, within or without a circle, draw its polar.

523. EXERCISE. Given a line, find its pole with respect to a given circle.

524. Exercise. If from a point without a circle two tangents are drawn to the circle, their chord of contact is the polar of the point.



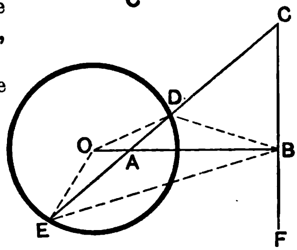
[To prove BC the polar of A.

OA is $\perp$ to BC. (?) $\triangle OBA$ is a R.A. $\triangle$. (?)

By Case III. of this Proposition,

$$\frac{OD}{OB}=\frac{OB}{OA}, \text{ or } OD \times OA = \overline{OB}^2.]$$

525. Exercise. Any line through the pole is divided harmonically by the pole, its polar, and the circumference.



[Let A be the pole of CF, and EC be any line through A.

To Prove $\frac{EA}{AD}=\frac{EC}{CD}$.

$$\frac{AO}{OD}=\frac{OD}{OB}. \ (?) \qquad \frac{AO}{OE}=\frac{OE}{OB}. \ (?)$$

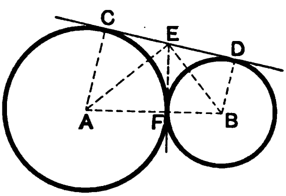
$\therefore$ $\triangle$s AOD and ODB are similar, as are also $\triangle$s OAE and OBE.

$$\frac{AD}{OD}=\frac{DB}{OB}. \ (?) \qquad \frac{OE}{AE}=\frac{OB}{EB}. \ (?) \qquad \frac{AD}{AE}=\frac{DB}{EB}. \ (?)$$

$\therefore$ BA bisects $\angle DBE$.

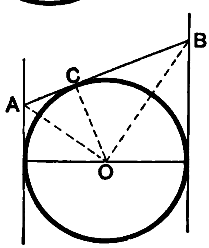
Since CB is $\perp$ to AB, CB bisects the exterior angle at B. Now apply § 510.]

526. Exercise. If two circles are tangent externally, the portion of their common tangent included between the points of contact is a mean proportional between the diameters of the circles.



[Show that AEB is a R.A. $\triangle$, and that EF (the half of CD) is a mean proportional between the radii.]

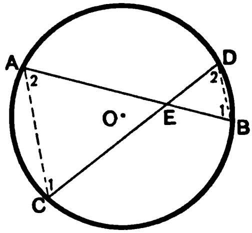
527. Exercise. If two tangents are drawn to a circle at the extremities of a diameter, the portion of any third tangent intercepted between them is divided at the point of contact into segments whose product is equal to the square of the radius.



[Show that OAB is a R.A. $\triangle$.]

PROPOSITION XXII. THEOREM

528. *If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.*



Let the chords AB and CD intersect at E.

To Prove $AE \cdot EB = CE \cdot ED$.

Proof. Draw AC and DB.

Prove $\triangle AEC$ and EDB mutually equiangular and therefore similar.

Whence $\frac{AE}{CE} = \frac{ED}{EB}$.

$\therefore AE \cdot EB = CE \cdot ED$. (?) Q.E.D.

CONVERSELY. If two lines AB and CD intersect at E, so that $AE \cdot EB = CE \cdot ED$, then can a circumference be passed through the four points A, B, C, and D.

[Pass a circumference through the three points A, B, and C. Then show that the point D cannot lie without this circumference, nor within it.]

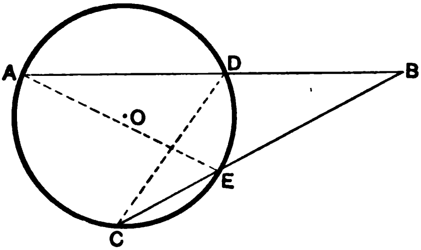
529. EXERCISE. C and D are respectively the middle points of a chord AB and its subtended arc. If AC is 8, and CD is 4, what is the radius of the circle?

530. EXERCISE. Two chords AB and CD intersect at the point E. AE is 8, EB is 6, and CD is 19. Find the segments of CD.

531. EXERCISE. If a chord is drawn through a fixed point within a circle, prove that the product of its segments is constant in whatever direction the chord is drawn.

PROPOSITION XXIII. THEOREM

532. *If from a point without a circle two secants be drawn terminating in the concave arc, the product of one secant and its external segment is equal to the product of the other secant and its external segment.*



Let AB and BC be two secants drawn from B to the circle whose center is O.

To Prove $AB \cdot DB = CB \cdot EB.$

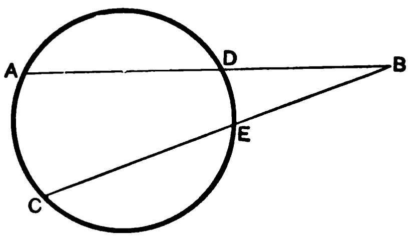
Proof. Draw AE and DC.

Prove △ AEB and CDB mutually equiangular and similar.

$$\frac{AB}{BC} = \frac{EB}{DB}. \quad (?)$$

$$\therefore AB \cdot DB = BC \cdot EB. \qquad \text{Q.E.D.}$$

CONVERSE. If on two intersecting lines AB and CB, four points, A, D, C, and E, be taken, so that $AB \times DB = BC \times EB$, then can a circumference be passed through the four points.

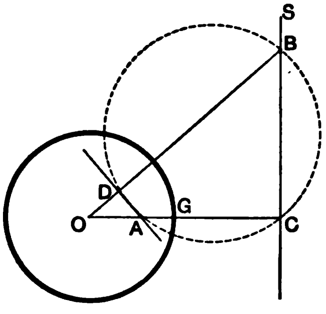


[Pass a circumference through three of the points, A, D, and E. Show by means of Prop. XXIII. and the hypothesis of the converse, that C can lie neither without nor within the circumference.]

533. Exercise. One of two secants meeting without a circle is 18 in., and its external segment is 4 in. long. The other secant is divided into two equal parts by the circumference. Find the length of the second secant.

534. Exercise. Two secants intersect without the circle. The external segment of the first is 5 ft., and the internal segment 19 ft. long. The internal segment of the second is 7 ft. long. Find the length of each secant.

535. Exercise. If A and B are two points such that the polar of A passes through B, then the polar of B passes through A.



Let CS, the polar of A, pass through B.

To Prove that the polar of B passes through A.

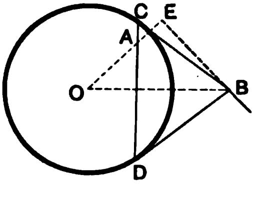
Proof. [Draw $AD \perp$ to OB.

The quadrilateral $ADBC$ has its opposite angles supplementary, $\therefore$ a circle can be circumscribed about it.

$$OD \times OB = OA \times OC = \overline{OG}^2.$$

$\therefore AD$ is the polar of B.]

536. Exercise. The locus of the intersection of tangents to a circle, at the extremities of any chord that passes through a given point, is the polar of the point.



Let CD be any chord passing through A, and B be the point of intersection of the tangents at C and D.

To Prove that B is a point of the polar of A.

[CD is the polar of B. (§ 524.)

The polar of B therefore passes through A.

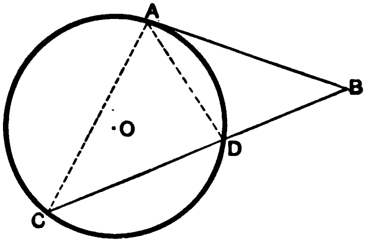
By § 535, the polar of A passes through B.]

537. Exercise. If from any point on a given line two tangents are drawn to a circle, their chord of contact passes through the pole of the line. [Apply § 535.]

538. Exercise. If from different points on a given straight line pairs of tangents are drawn to a circle, their chords of contact all pass through a common point.

PROPOSITION XXIV. THEOREM.

539. ***If from a point without a circle a secant and a tangent are drawn, the secant terminating in the concave arc, the square of the tangent is equal to the product of the secant and its external segment.***



Let AB be a tangent and BC a secant drawn from B to the circle whose center is O.

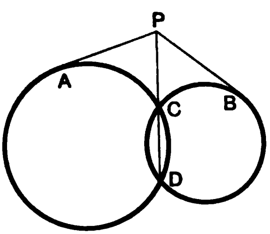
To Prove $\overline{AB}^2 = BC \times DB$.

Proof. Draw AC and AD.

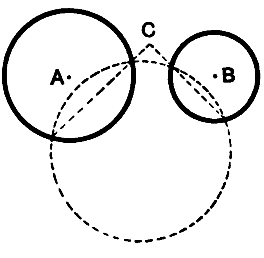
Prove $\triangle$ CAB and DAB similar.

Whence $\dfrac{BC}{AB} = \dfrac{AB}{DB}$.

$\therefore \overline{AB}^2 = BC \times DB$. Q.E.D.



540. EXERCISE. Tangents drawn to two intersecting circles from a point on their common chord produced, are equal.



541. EXERCISE. Given two circles, to find a point such that the tangents drawn from it to the two circles are equal.

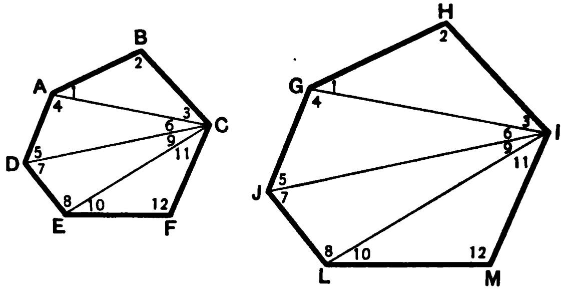
[Describe any circle intersecting the two given circles.

Draw the two common chords.

Prove that tangents drawn to the two circles from C, the point of intersection of the common chords (prolonged), are equal.]

PROPOSITION XXV. THEOREM

542. *Two polygons are similar if they are composed of the same number of triangles, similar each to each, and similarly placed.*



Let the $\triangle$ ABC, ADC, DEC, and EFC be similar respectively to the $\triangle$ GHI, GJI, JLI, and LMI, and be similarly placed.

To Prove polygons $ABCFED$ and $GHIMLJ$ similar.

Proof. Show that the angles of $ABCFED$ are equal respectively to the corresponding angles of $GHIMLJ$.

$$\frac{AB}{GH} = \frac{AC}{GI}. \quad (?)$$

$$\frac{AD}{GJ} = \frac{AC}{GI}. \quad (?)$$

Whence $$\frac{AB}{GH} = \frac{AD}{GJ}. \quad (?)$$

Similarly prove $$\frac{AD}{GJ} = \frac{DE}{JL}, \text{ etc.}$$

$$\therefore \frac{AB}{GH} = \frac{AD}{GJ} = \frac{DE}{JL} = \frac{EF}{LM} = \frac{FC}{MI} = \frac{CB}{IH}.$$

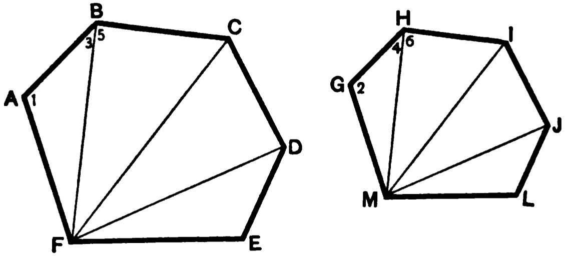
The polygons are mutually equiangular and have their corresponding sides proportional. They are therefore similar by definition. Q.E.D.

543. Corollary. *On a given line as a side to construct a polygon similar to a given polygon.*

544. Definition. In similar polygons the corresponding sides are called *homologous sides*, and the equal angles are called *homologous angles*.

Proposition XXVI. Theorem

545. ***Two similar polygons can be divided into the same number of similar triangles, similarly placed.***



Let $ABCDEF$ and $GHIJLM$ be two similar polygons.

To Prove that they can be divided into the same number of similar triangles, similarly placed.

Proof. From the vertex F draw all the possible diagonals.

From M, homologous with F, draw all the possible diagonals.

Prove △ FAB and MGH similar (§ 495).

Whence $\angle 3 = \angle 4$.

$\angle 5 = \angle 6$. (?)

$$\frac{AB}{GH} = \frac{BF}{HM}. \quad (?) \qquad \frac{AB}{GH} = \frac{BC}{HI}. \quad (?)$$

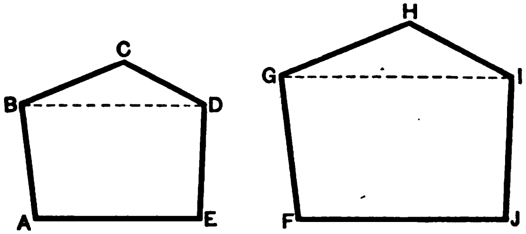
$$\frac{BF}{HM} = \frac{BC}{HI}. \quad (?)$$

△ FBC and MHI are similar. (?)

Show that △ FCD and MIJ are similar, and also △ FDE and MJL. Q.E.D.

PROPOSITION XXVII. THEOREM

546. *The perimeters of similar polygons are to each other as any two homologous sides.*



Let $ABCDE$ and $FGHIJ$ be two similar polygons.

To Prove $$\frac{AB + BC + CD + \text{etc.}}{FG + GH + HI + \text{etc.}} = \frac{CD}{HI}.$$

Proof. By definition

$$\frac{AB}{GF} = \frac{BC}{GH} = \frac{CD}{HI} = \frac{DE}{IJ} = \frac{AE}{FJ}.$$

[Apply § 443.]

547. COROLLARY. *The perimeters of similar polygons are to each other as any two homologous diagonals.*

548. EXERCISE. The perimeters of similar triangles are to each other as any homologous altitudes.

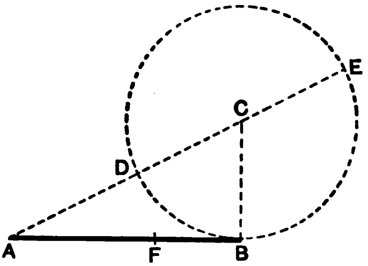
549. EXERCISE. The perimeters of similar triangles are to each other as any homologous medians.

550. EXERCISE. The perimeters of two similar polygons are 78 and 65; a side of the first is 9, find the homologous side of the second.

551. DEFINITION. A line is divided in *extreme and mean ratio* when it is divided into two parts so that one segment is a mean proportional between the whole line and the other segment.

PROPOSITION XXVIII. PROBLEM

552. *To divide a line in extreme and mean ratio.*



Let AB be the given line.

Required to divide AB in extreme and mean ratio.

Draw $BC \perp$ to AB and equal to one half of AB. Draw AC. With C as a center and CB as a radius describe a circle cutting AC at D, and AC prolonged at E. Lay off $AF = AD$.

$$\frac{AE}{AB} = \frac{AB}{AD}. \quad (\S\ 539.) \qquad \frac{AE - AB}{AB} = \frac{AB - AD}{AD}. \quad (?)$$

$$\frac{AD}{AB} = \frac{AB - AF}{AD}. \quad (?) \qquad \frac{AF}{AB} = \frac{FB}{AF}. \quad (?) \qquad \frac{AB}{AF} = \frac{AF}{FB}. \quad (?)$$

Q.E.F.

553. Exercise. To determine the values of the segments of a line that has been divided in extreme and mean ratio.

In the figure of § 552, let the length of AB be a; $AF = x$, then $FB = a - x$.

Substituting these values in the last proportion, we get

$$\frac{a}{x} = \frac{x}{a - x}, \text{ whence } a^2 - ax = x^2.$$

Solving the equation,

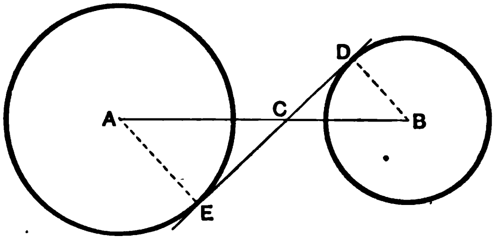
$$x = \tfrac{1}{2}a\sqrt{5} - \tfrac{1}{2}a = \frac{a}{2}(\sqrt{5} - 1),$$

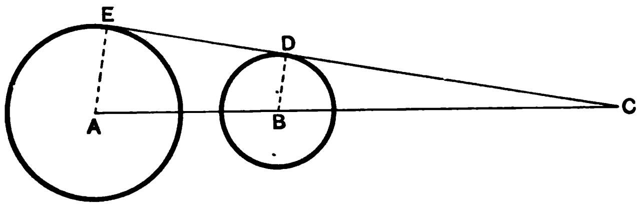
$$a - x = \tfrac{3}{2}a - \tfrac{1}{2}a\sqrt{5} = \frac{a}{2}(3 - \sqrt{5}).$$

554. Exercise. Divide a line 5 in. long in extreme and mean ratio, and calculate the value of the segments.

PROPOSITION XXIX. PROBLEM

555. *To draw a common tangent to two given circles.*





Let A and B be the centers of the two given circles.

Required to draw a common tangent to the two circles.

Let R stand for the radius of circle A, and r for the radius of circle B.

Draw AB. Divide AB (internally and externally) at C so that $\frac{AC}{BC}=\frac{R}{r}$.

Draw CD tangent to circle B. Draw the radius BD.

Draw $AE \perp$ to DC prolonged.

[It is required to show that $AE = R$.]

$\triangle$ AEC and CBD are similar (?), whence $\frac{AC}{BC}=\frac{AE}{BD}$.

$\therefore$ $\frac{R}{r}=\frac{AE}{r}$, and $AE = R$, and ED is a common tangent. Q.E.F.

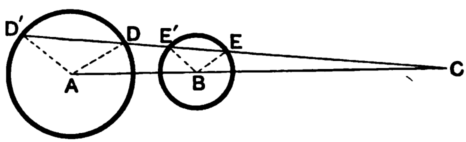
556. DEFINITION. The two tangents that pass through the internal point of division of AB are called the *transverse* tangents. The two tangents that pass through the external point of division are called the *direct* tangents.

The points of division are called the *centers of similitude* of the two circles.

557. Exercise. The line joining the centers of two circles is divided harmonically by the centers of similitude.

558. Exercise. The line joining the extremities of parallel radii of two circles passes through their external center of similitude if the radii are turned in the same direction; but through their internal center if they are turned in opposite directions.

559. Exercise. All lines passing through a center of similitude of two circles and intersecting the circles are divided by the circumferences in the same ratio.



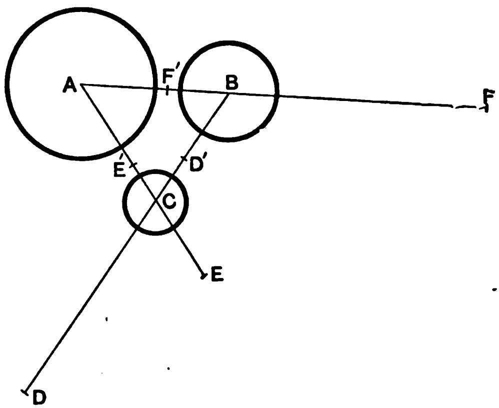
Draw the *radius* AD.

Draw a *line* BE parallel to AD, and by means of similar triangles prove that BE is a radius. Then

$$\frac{CE}{CD}=\frac{r}{R}.$$

Similarly,

$$\frac{CE'}{CD'}=\frac{r}{R}.$$



560. Exercise. A, B, and C are the centers of three circles; a, b, and c their respective radii; D, E, and F their external centers of similitude; and D', E', and F' their internal centers of similitude.

Prove that D, E, and F are in a straight line.

$$\left[\frac{AF}{FB}=\frac{a}{b},\ \frac{BD}{DC}=\frac{b}{c},\ \text{and}\ \frac{CE}{EA}=\frac{c}{a},\ \text{whence}\ \frac{AF}{FB}\times\frac{BD}{DC}\times\frac{CE}{EA}=1.\right]$$

Similarly, show that D, E', and F' are in a straight line, also E, D', and F', and also F, D', and E'.

EXERCISES

1. If
$$\frac{a}{b}=\frac{c}{d},$$

Prove $\frac{b-a}{a}=\frac{d-c}{c}$, $\frac{b-a}{b}=\frac{d-c}{d}$, $\frac{a}{b}=\frac{c-a}{d-b}$, $\frac{a}{3a+b}=\frac{c}{3c+d}$.

2. If
$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f},$$
prove
$$\frac{xa-ye+zc}{xb-yf+zd}=\frac{e}{f}.$$

3. If
$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h},$$
prove
$$\frac{ac+eg}{bd+fh}=\frac{c^2}{d^2}.$$

4. If
$$\frac{a}{b}=\frac{b}{c},$$
prove
$$\frac{a^2+ab}{b^2+bc}=\frac{a}{c}.$$

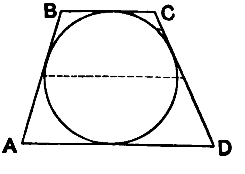
5. If
$$\frac{a}{b}=\frac{b}{c},$$
prove
$$\frac{a}{c}=\frac{(a+b)^2}{(b+c)^2}.$$

6. If
$$\frac{a}{x^2}=\frac{b}{y^2}=\frac{c}{z^2}, \text{ and } a+b=c,$$
prove
$$x^2+y^2=z^2.$$

7. The shadow cast by a church steeple on level ground is 27 yd., while that cast by a 5-ft. vertical rod is 3 ft. long. How high is the steeple?

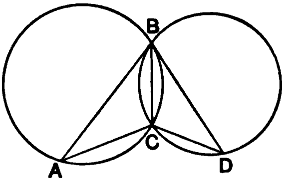
8. The line joining the middle points of the non-parallel sides of a trapezoid circumscribed about a circle is equal to one fourth the perimeter of the trapezoid.

[See § 396.]



9. Two circles intersect at B and C. BA and BD are drawn tangent to the circles.

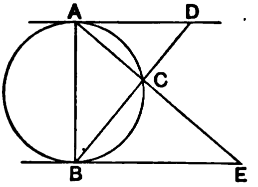
Prove that BC is a mean proportional between AC and CD. [Prove ▲ ABC and BCD similar.]



10. Find the lengths of the longest and the shortest chords that can be drawn through a point 10 in. from the center of a circle having a radius 26 in.

11. Tangents are drawn to a circle at the extremities of the diameter AB. Secants are drawn from A and B, meeting the tangents at D and E and intersecting at C on the circumference.

Prove the diameter a mean proportional between the tangents AD and BE. [▲ ABD and ABE are similar. (?)]



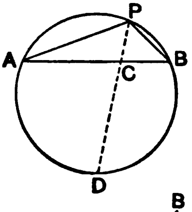
12. If two circles are tangent internally, chords of the greater drawn from the point of tangency are divided proportionally by the circumference of the less.

13. If two circles are tangent externally, secants drawn through their point of contact and terminating in the circumferences are divided proportionally at the point of contact.

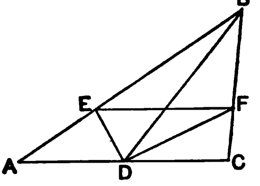
14. Given the two segments of the base of a triangle made by the bisector of the vertical angle, and the sum of the other two sides, to construct the triangle. [§ 502.]

15. Determine a point P in the circumference, from which chords drawn to two given points A and B shall have the ratio $\frac{m}{n}$.

[Divide AB so that $\frac{AC}{CB}=\frac{m}{n}$. Join C with the middle point of the arc ADB.]

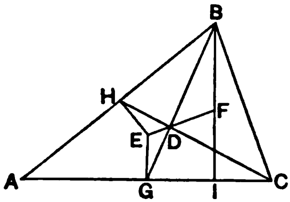


16. In the triangle ABC, BD is a medial line, and DE and DF bisect angles ADB and BDC respectively. Prove that EF is parallel to AC.



17. D is the point of intersection of the medians; E is the point of intersection of the perpendiculars at the middle points of the sides; DE is prolonged to meet the altitude BI at F. Prove $ED = \frac{1}{2} DF$.

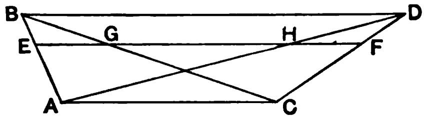
[$\triangle$ EDG and DBF are similar, and $BD = 2\,DG$.]



18. The point of intersection of the medians, the point of intersection of the perpendiculars at the middle points of the sides, and the point of intersection of the altitudes of a triangle are in the same straight line. [See Ex. 17.]

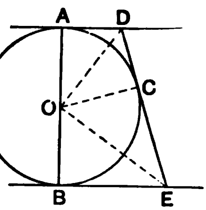
19. The triangles ABC and ADC have the same base and lie between the same parallels. EF is drawn parallel to AC.

Prove $EG = HF$.

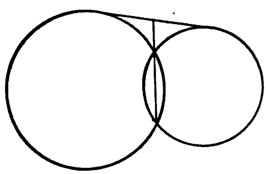


20. Two tangents are drawn at the extremities of the diameter AB. At any other point C on the circumference a third tangent DE is drawn. Prove that OD is a mean proportional between AD and DE, and that OE is a mean proportional between BE and DE.

[Prove $\angle DOE$ a R.A., and use § 518.]

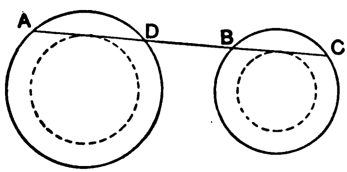


21. The prolongation of the common chord of two intersecting circles bisects their common tangent. [§ 539.]

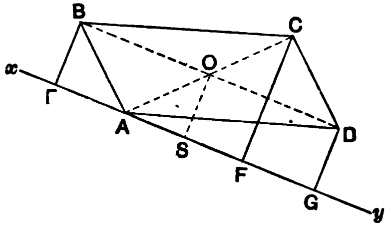


22. To draw a line AC intersecting two given circles so that the chords AD and BC shall be of given lengths.

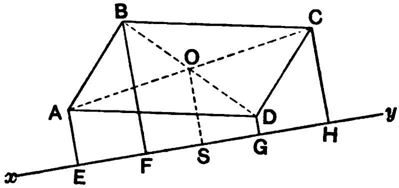
[See Ex. 24, p. 125.]



23. xy is any line drawn through the vertex A of the parallelogram $ABCD$ and lying without the parallelogram. Prove that the perpendicular to xy from the opposite angle C is equal to the sum of the perpendiculars from B and D to xy. [§ 453.]

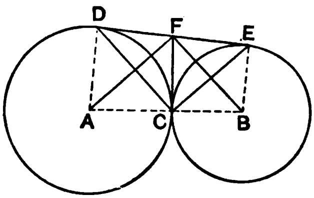


24. The sum of the perpendiculars from the vertices of one pair of opposite angles to a line lying without a parallelogram is equal to the sum of the perpendiculars from the vertices of the other pair of opposite angles.

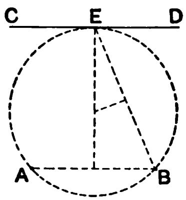


25. Two circles are tangent externally at C. DE and CF are common tangents. Prove that $\angle DCE = 1$ R.A., and also that $\angle AFB = 1$ R.A.

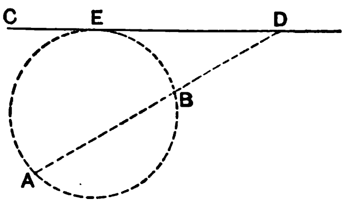
26. Prove that ▲ DFC and CBE (see figure of Ex. 25) are similar, as are also ▲ DAC and FCE.



27. Describe a circle passing through two given points and tangent to a given line.



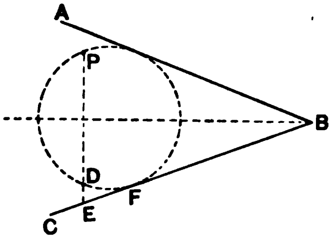
(1)



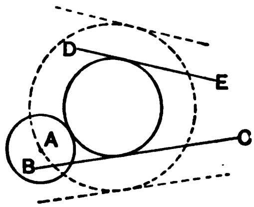
(2)

[The line joining the two given points A and B may be parallel to the given line CD (see Fig. 1), or its prolongation may meet the given line (see Fig. 2). In the second case $DE^2 = DA \times DB$. (?) DE may be laid off on *either* side of D, ∴ two Ⓢ can be described fulfilling the conditions of the problem.]

28. Describe a circle tangent to two given lines and passing through a given point. [P is the given point. Find another point D through which the circumference must pass. Then solve as in Ex. 27.]

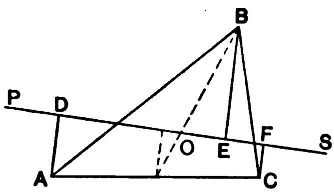


29. Describe a circle tangent to two given lines and tangent to a given circle. [DE and BC are the lines, and A the center of the given circle. Use Ex. 28.]

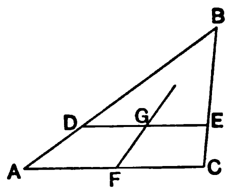


30. Through a given point P draw a line cutting a triangle so that the sum of the perpendiculars to the line from the two vertices on one side of the line shall equal the perpendicular from the vertex on the other side of the line.

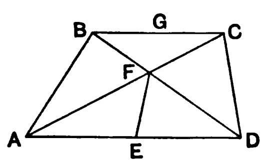
[O is the point of intersection of the medians.]

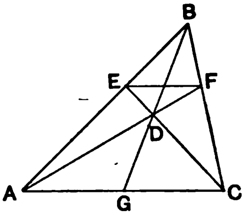


31. In the triangle ABC, DE is drawn parallel to AC. FG connects the middle points of AC and DE. Prove that FG prolonged passes through B.

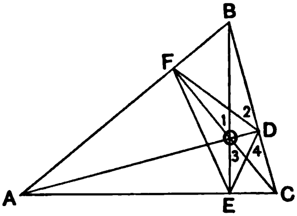


32. The line joining the middle point of the lower base of a trapezoid with the point of intersection of the diagonals bisects the upper base.





33. In the triangle ABC, let two lines drawn from the extremities of the base AC and intersecting at any point D on the median through B, meet the opposite sides in E and F. Show that EF is parallel to AC.

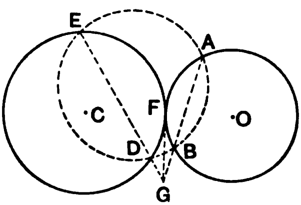


34. ABC is an acute-angled triangle. DEF (called the *pedal triangle*) is formed by joining the feet of the altitudes of triangle ABC. Prove that the altitudes of triangle ABC bisect the angles of the pedal triangle DEF. [A ⊙ can be described passing through F, O, D, and B. (?) $\angle 1 = \angle 2$. (?)]

35. Prove the triangles AFE, BFD, and DCE similar to triangle ABC and to each other. [See figure of Ex. 34.] [To prove ⊿s FBD and ABC similar. Show that $\angle A = \angle 2$.]

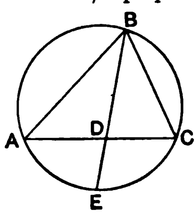
36. Prove that the sides of the triangle ABC [see Ex. 34] bisect the exterior angles of the pedal triangle DEF.

37. The three circles that pass through two vertices of a triangle and the point of intersection of the altitudes are equal to each other. [Show that each is equal to the circle circumscribed about the triangle.]



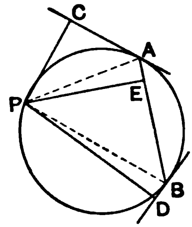
38. Describe a circle passing through two given points and tangent to a given circle. [A and B are the given points and C the given circle. $DEAB$ is any ⊙ passing through A and B and cutting the given ⊙ C. The common chord ED meets AB at G. GF is tangent to ⊙ C. AFB is the required ⊙.]

39. If one leg of a right-angled triangle is double the other, a perpendicular from the right angle to the hypotenuse divides it into segments having the ratio of 1 to 4.

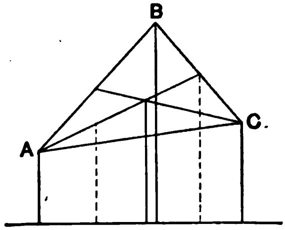


40. The triangle ABC is inscribed in a circle, and the bisector of angle B intersects AC at D and the circumference at E. Prove

$$\frac{AB}{BD} = \frac{BE}{BC}.$$

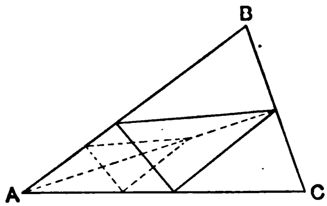


41. The perpendicular drawn to a chord from any point in the circumference is a mean proportional between the perpendiculars from that point to the tangents drawn at the extremities of the chord.

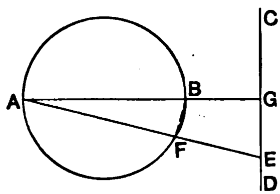


42. The perpendicular drawn from the point of intersection of the medians of a triangle to a line without the triangle is equal to one third the sum of the perpendiculars from the vertices of the triangle to that line. [§ 453.]

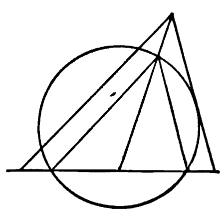
43. Construct a right-angled triangle, having given an acute angle and the perimeter.



44. Inscribe in a given triangle another triangle, the sides of which are parallel to the sides of a second given triangle.



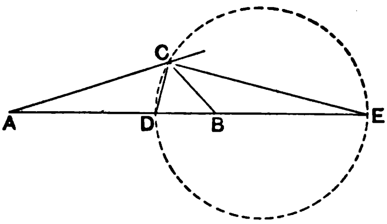
45. CD is a line perpendicular to the diameter AB. AE is drawn from A to *any* point on CD. Prove that $AE \times AF$ is constant. [A circle can be passed through F, B, G, and E. (?)]



46. Given the vertical angle, the medial line to the base, and the angle that the medial line makes with the base, to construct the triangle.

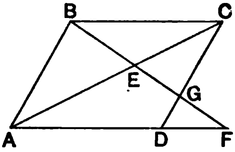
47. Given the base of a triangle and the ratio of the other two sides, to find the locus of its vertex.

[Divide the given base AB harmonically at D and E, in the ratio of the two given sides. On DE as a diameter construct a ⊙.]



48. In the parallelogram $ABCD$, BF is drawn cutting the diagonal AC in E, CD in G, and AD prolonged in F.

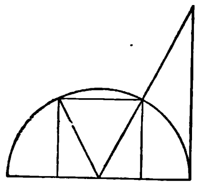
Prove that $\overline{BE}^2 = GE \times EF$.



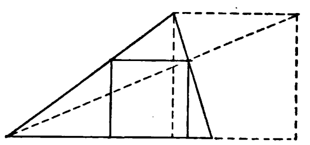
49. If three circles intersect each other, their common chords intersect in the same point. [§ 528.]

50. In any inscribed quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides.

51. To inscribe a square in a given semicircle.



52. To inscribe a square in a given triangle.



53. $ABCD$ is a parallelogram, E a point on BC such that BE is one fourth of BC. AE cuts the diagonal BD in F. Show that BF is one fifth of BD.

54. Two chords of a circle drawn from a common point A on the circumference and cut by a line parallel to a tangent through A, are divided proportionally. [*Suggestion.* Join the extremities of the chords and prove the triangles similar.]

BOOK IV

561. Definitions. We *measure* a magnitude by comparing it with a similar magnitude that is taken as the unit of measure. If we wish to find the length of a line, we find how many times a linear unit of measure, say a foot, is contained in the line. This number, with the proper denomination, is called the length of the line.

Similarly, we measure any portion of a surface by comparing it with some unit of surface measure. We find how many times this unit, say a square yard, is contained in the portion of surface. This number, with the denomination square yards, we call the *area* or *superficial content* of the surface measured.

Polygons that have the same areas are *equivalent polygons*. Equivalent polygons are not necessarily equal in all respects. They need not even have the same number of sides. For example, a triangle, a square, and a hexagon may be equivalent.

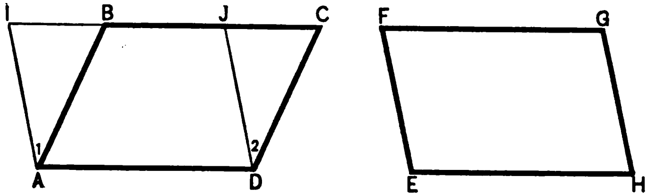
The *base* of a polygon is primarily the side upon which the figure *stands*; but usage has sanctioned a more extended application of the term. Any side of a polygon may be considered the base. In a parallelogram, if two opposite sides are horizontal lines, they are frequently called the *upper and lower bases* of the parallelogram. In a trapezoid, the two parallel sides are called its bases.

The *altitude* of a parallelogram is the perpendicular distance between two opposite sides. A parallelogram may therefore have two different altitudes.

The *altitude* of a trapezoid is the perpendicular distance between its bases.

PROPOSITION I. THEOREM

562. *Parallelograms having equal bases and equal altitudes are equivalent.*



Let $ABCD$ and $EFGH$ be two parallelograms having equal bases and equal altitudes.

To Prove $ABCD$ and $EFGH$ equal in area.

Proof. Place $EFGH$ upon $ABCD$ so that their lower bases shall coincide. Because they have equal altitudes their upper bases are in the same line.

Prove $\triangle s$ AIB and DJC equal.

The parallelogram $AIJD$ is composed of the quadrilateral $ABJD$ and the $\triangle AIB$.

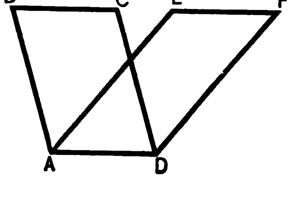
The parallelogram $ABCD$ is composed of the quadrilateral $ABJD$ and the $\triangle DJC$.

$$ABCD = AIJD.$$

$$ABCD = EFGH.$$ Q.E.D.

563. EXERCISE. Rectangles having equal bases and altitudes are equal in all respects.

564. EXERCISE. Construct a rectangle equivalent to a given parallelogram.



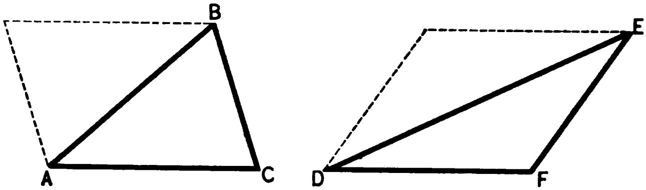
565. EXERCISE. Prove Prop. I., using this figure:

566. EXERCISE. Construct a rectangle whose area is double that of a given equilateral triangle.

567. Exercise. A line joining the middle points of two opposite sides of a parallelogram divides the figure into two equivalent parallelograms.

Proposition II. Theorem

568. *Triangles having equal bases and equal altitudes are equivalent.*



Let the ⊿ *ABC* and *DEF* have equal bases and equal altitudes.

To Prove the △ *ABC* and *DEF* equal in area.

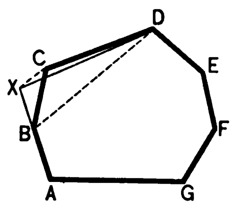
Proof. On each triangle construct a parallelogram having for its base and altitude the base and altitude of the triangle.

These parallelograms are equivalent. (?)

∴ the triangles are equivalent. (?) Q.E.D.

569. Corollary I. *If a triangle and a parallelogram have equal bases and equal altitudes, the triangle is equivalent to one half the parallelogram.*

570. Corollary II. *To construct a triangle equivalent to a given polygon.*



To construct a triangle equivalent to *ABC . . . G*.

Draw *BD*.

Through *C* draw *CX* parallel to *BD*, meeting *AB* prolonged at *X*.

Draw *DX*.

Show that ⊿ *BXD* and *BCD* have a common base and equal altitudes. ∴ △ *BXD* = △ *BCD*, and the polygon *AXDEFG* = polygon *ABCDEFG*.

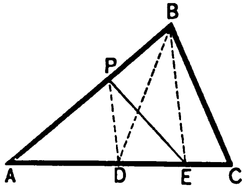
We have therefore constructed a polygon equivalent to the given polygon and having *one side less* than the given polygon has. A new polygon may be constructed equivalent to this polygon and having one side less; and this process can be repeated until a triangle is reached.

571. Exercise. Two triangles are equivalent if they have two sides of the one equal respectively to two sides of the other, and the included angles supplementary. [Place the ⩓ so that the two supplementary ∡ are adjacent and a side of one △ coincides with its equal in the other.]

572. Exercise. Bisect a triangle by a line drawn from a vertex.

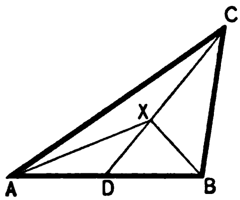
573. Exercise. Bisect a triangle by a line drawn from a point in the perimeter.

[*BD* is a medial line, *BE* is drawn ∥ to *PD*. Show that *PE* bisects △ *ABC*.]



574. Exercise. The diagonals of a parallelogram divide it into four equivalent triangles.

575. Exercise. The three medial lines of a triangle divide it into six equivalent triangles.



576. Exercise. In the triangle *ABC*, *X* is any point on the median *CD*. Prove that the triangles *AXC* and *BXC* are equivalent.

577. Exercise. On the base of a given triangle construct a second triangle equal in area to the first, and having its vertex in a given straight line. Under what conditions is this exercise impossible?

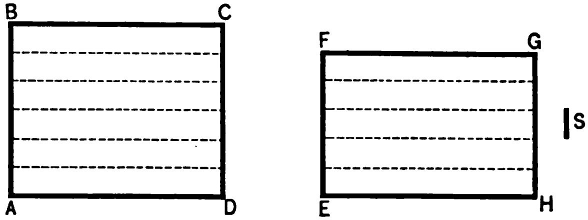
578. Exercise. Construct a right-angled triangle equivalent to a given equilateral triangle.

579. Exercise. From a point in the perimeter of a parallelogram draw a line that shall divide the parallelogram into two equivalent parts.

580. Exercise. Construct an isosceles triangle equivalent to a given square.

PROPOSITION III. THEOREM

581. *Rectangles having equal bases are to each other as their altitudes.*



CASE I. When the altitudes are commensurable.

Let $ABCD$ and $EFGH$ be rectangles having equal bases and commensurable altitudes.

To Prove $$\frac{ABCD}{EFGH}=\frac{AB}{EF}.$$

Proof. Let S be the unit of measure for the altitudes, and let it be contained in AB m times and in EF n times, whence

$$\frac{AB}{EF}=\frac{m}{n}. \qquad (1)$$

Divide the altitudes by the unit of measure and through the points of division draw parallels to the bases.

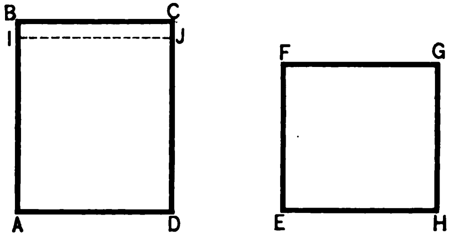
$ABCD$ is divided into m parallelograms and $EFGH$ into n parallelograms, and these parallelograms are all equal by § 562.

$$\therefore \quad \frac{ABCD}{EFGH}=\frac{m}{n}. \qquad (2)$$

Apply Axiom 1 to (1) and (2).

$$\frac{ABCD}{EFGH}=\frac{AB}{EF}.$$ Q.E.D.

CASE II. When the altitudes are incommensurable.



Let the parallelograms $ABCD$ and $EFGH$ have equal bases and incommensurable altitudes.

To Prove $$\frac{ABCD}{EFGH}=\frac{AB}{EF}.$$

Proof. Let EF be divided into a number of equal parts, and let one of these parts be applied to AB as a unit of measure.

Since AB and EF are incommensurable, AB will not contain the unit of measure exactly, but a certain number of these parts will extend as far as I, leaving a remainder IB smaller than the unit of measure.

Through I draw IJ parallel to the base AD.

$$\frac{AIJD}{EFGH}=\frac{AI}{EF} \text{ by Case I.}$$

By increasing indefinitely the number of equal parts into which EF is divided, the divisions will become smaller and smaller, and the remainder IB will also diminish indefinitely.

Now $\frac{AIJD}{EFGH}$ is evidently a variable, as is also $\frac{AI}{EF}$; and these variables are always equal. (Case I.)

The limit of the variable $\frac{AIJD}{EFGH}$ is $\frac{ABCD}{EFGH}$.

The limit of the variable $\frac{AI}{EF}$ is $\frac{AB}{EF}$.

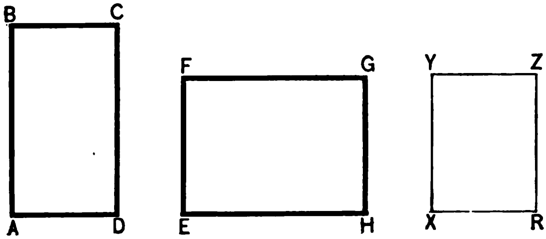
By § 341 $$\frac{ABCD}{EFGH}=\frac{AB}{EF}.$$ Q.E.D

582. Corollary. *Rectangles having equal altitudes are to each other as their bases.*

583. Exercise. The altitudes of two rectangles having equal bases are 12 ft. and 16 ft. respectively. The area of the former rectangle is 96 sq. ft. What is the area of the other?

Proposition IV. Theorem

584. *Any two rectangles are to each other as the products of their bases and altitudes.*



Let $ABCD$ and $EFGH$ be any two rectangles.

To Prove $$\frac{ABCD}{EFGH} = \frac{AD \times AB}{EH \times EF}.$$

Proof. Construct a third rectangle $XYZR$, having a base equal to the base of $ABCD$ and an altitude equal to the altitude of $EFGH$.

$$\frac{ABCD}{XYZR} = \frac{AB}{XY}. \qquad (?)$$

$$\frac{EFGH}{XYZR} = \frac{EH}{XR}. \qquad (?)$$

$$\frac{ABCD}{EFGH} = \frac{AB \times XR}{XY \times EH}. \qquad (?)$$

$$\frac{ABCD}{EFGH} = \frac{AB \times AD}{EF \times EH}. \qquad (?)$$

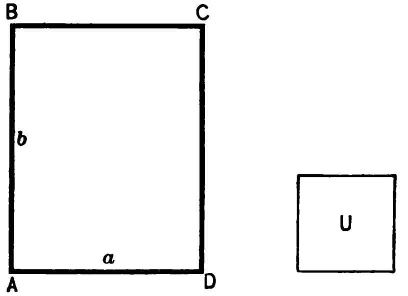
Q.E.D.

585. Exercise. The base and the altitude of a certain rectangle are 5 ft. and 4 ft. respectively. The base and the altitude of a second rectangle are 10 ft. and 8 ft. respectively. How do their areas compare?

[The student must not *assume* that the area of the first rectangle is 20 sq. ft., as that has not yet been established.]

Proposition V. Theorem

586. *The area of a rectangle is equal to the product of its base and altitude.*



Let $ABCD$ be any rectangle.

To Prove $ABCD = a \times b$.

Proof. Let the square U, each side of which is a linear unit, be the unit of measure for surfaces.

$$\frac{ABCD}{U} = \frac{a \times b}{1 \times 1}. \quad (?)$$

Whence $ABCD = ab \times U$.

or $ABCD = ab \times$ the surface unit.

or $ABCD = ab$ surface units.

This is usually abbreviated into

$$ABCD = a \times b. \quad (1)$$

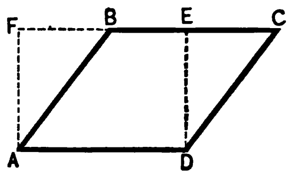
Q.E.D.

587. SCHOLIUM. The meaning to be attached to formula (1) is, that the number of surface units in a rectangle is the same as the product of the number of linear units in the base by the number of linear units in the altitude.

If the base is 4 ft. and the altitude 3 ft., the number of square feet (surface units) in the rectangle is 4×3 or 12.

The area then is 12 square feet.

588. COROLLARY I. *The area of any parallelogram is equal to the product of its base and altitude.*



Let $ABCD$ be any parallelogram and DE be its altitude.

To Prove $ABCD = AD \times DE$.

Proof. Draw $AF \perp$ to AD, meeting BC prolonged at F.

Prove $ADEF$ a rectangle.

$$ADEF = ADCB. \quad (?)$$

$$ADEF = AD \times ED. \quad (?)$$

$$ABCD = AD \times ED. \quad (?)$$

Q.E.D.

589. COROLLARY II. *Any two parallelograms are to each other as the products of their bases and altitudes; if their bases are equal the parallelograms are to each other as their altitudes; if the altitudes are equal the parallelograms are to each other as their bases.*

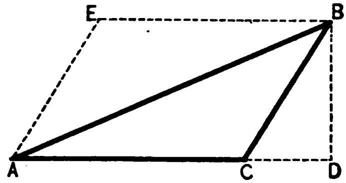
590. EXERCISE. Construct a square equivalent to a given parallelogram.

591. EXERCISE. Construct a rectangle having a given base and equivalent to a given parallelogram.

592. EXERCISE. Of all equivalent parallelograms having a common base, the rectangle has the least perimeter. Of all equivalent rectangles, the square has the least perimeter.

Proposition VI. Theorem

593. *The area of a triangle is one half the product of its base and altitude.*



Let ABC be any △, and BD its altitude.

To Prove $ABC = \frac{1}{2} AC \times BD.$

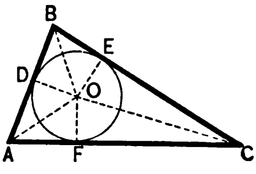
Proof. Construct the parallelogram $ACBE$.

$$ACBE = AC \times BD. \quad (?)$$

$$\triangle ABC = \frac{1}{2} AC \times BD. \quad (?)$$ Q.E.D.

594. Corollary I. *Triangles are to each other as the products of their bases and altitudes; if their bases are equal the triangles are to each other as their altitudes; if their altitudes are equal the triangles are to each other as their bases.*

595. Corollary II. *The area of a triangle is one half the product of its perimeter and the radius of the inscribed circle.*



[Draw radii to the points of tangency.

Connect the center O with the three vertices.

Show that OD is the altitude of △ AOB, and that OE and OF are altitudes of △s BOC and AOC. Call the radius of the inscribed circle r.

$$\triangle AOB = \frac{1}{2} AB \cdot r. \quad (?)$$

$$\triangle BOC = \frac{1}{2} BC \cdot r. \quad (?)$$

$$\triangle AOC = \frac{1}{2} AC \cdot r. \quad (?)$$

$$\triangle ABC = \frac{1}{2} (AB + BC + CA) r. \quad (?)$$ Q.E.D.]

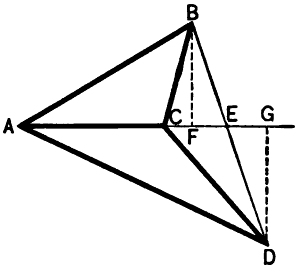
596. Corollary III. *Calling $2s$ the perimeter of the triangle ABC, $\triangle ABC = s\,r$, whence $r = \frac{\triangle ABC}{s}$. The radius of the inscribed circle of a triangle is equal to the area of the triangle divided by one half its perimeter.*

597. Exercise. The area of a rhombus is equal to one half the product of its diagonals.

598. Exercise. Construct a square equivalent to a given triangle.

599. Exercise. Construct a square equivalent to a given polygon.

600. Exercise. Two triangles having a common base are to each other as the segments into which the line joining their vertices is divided by the common base, or base produced.



[The $\triangle$s ABC and ACD have the common base AC; to prove

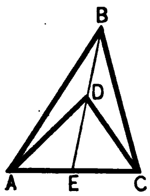
$$\frac{\triangle ABC}{\triangle ADC} = \frac{BE}{ED}.$$

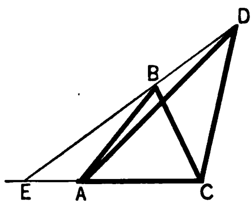
Draw the altitudes BF and DG.

$$\frac{BE}{ED} = \frac{BF}{DG}. \quad (?) \qquad \frac{\triangle ABC}{\triangle ADC} = \frac{BF}{DG}. \quad (?)$$

$$\frac{\triangle ABC}{\triangle ADC} = \frac{BE}{ED}. \qquad \text{Q.E.D.}]$$

Note. When the two triangles are on the same side of the common base, BD, the line joining their vertices is divided *externally* at E.

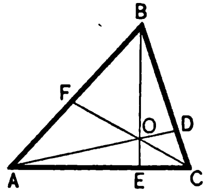




Prove $\frac{\triangle ABC}{\triangle ADC} = \frac{BE}{DE}$, using these figures.

601. Definition. Lines that pass through a common point are called *concurrent lines.*

602. Exercise. If three concurrent lines AO, BO, and CO, drawn from the vertices of the triangle ABC, meet the opposite sides in the points D, E, and F, prove $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1.$

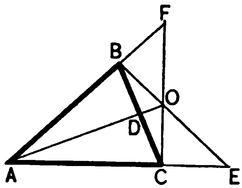


[The point O may be within or without the triangle.

$\frac{BD}{DC} = \frac{\triangle AOB}{\triangle AOC}.$ (?) $\frac{CE}{EA} = \frac{\triangle BOC}{\triangle AOB}.$ (?)

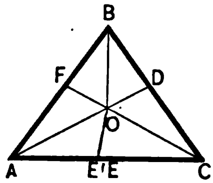
$$\frac{AF}{FB} = \frac{\triangle AOC}{\triangle BOC}. \quad (?)$$

$$\therefore \frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1.]$$



Conversely, if $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1$, to prove that the lines AD, BE, and CF are concurrent.

[Draw AD and CF. Call their point of intersection O. Draw BO. Suppose BO prolonged *does not go to* E, but some other point of AC, as E'.



$$\frac{BD}{DC} \times \frac{CE'}{E'A} \times \frac{AF}{FB} = 1. \quad (?)$$

$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1. \quad \text{(Hypothesis.)}$$

$$\frac{CE'}{E'A} = \frac{CE}{EA}. \quad (?)$$

Show that this last proportion is absurd.

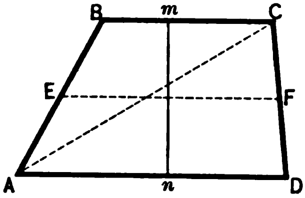
$\therefore$ AD, BE, and CF are concurrent.]

603. Exercise. Show by means of the converse of the last exercise that the following lines in a triangle are concurrent.

1. The medial lines.
2. The bisectors of the angles.
3. The altitudes.

PROPOSITION VII. THEOREM

604. *The area of a trapezoid is one half the product of its altitude and the sum of its parallel sides.*



Let $ABCD$ be a trapezoid, and mn be its altitude.

To Prove $ABCD = \frac{1}{2}\, mn(BC + AD)$.

Proof. Draw the diagonal AC.

Show that mn is equal to the altitude of each triangle formed.

$\triangle ABC = \frac{1}{2}\, mn \cdot BC.$ (?)

$\triangle ACD = \frac{1}{2}\, mn \cdot AD.$ (?)

$ABCD = \frac{1}{2}\, mn(BC + AD).$ (?) Q.E.D.

605. COROLLARY. *The area of a trapezoid is equal to the product of the altitude and the line joining the middle points of the non-parallel sides.*

[$EF = \frac{1}{2}(BC + AD)$ (?) $\therefore\ ABCD = mn \cdot EF.$]

606. EXERCISE. In the figure for § 604 let $BC = 8$ in., $AD = 12$ in., and $mn = EF$. Find the area of the trapezoid.

607. EXERCISE. Construct a square equivalent to a given trapezoid.

608. EXERCISE. Construct a rectangle equivalent to a given trapezoid and having its altitude equal to that of the trapezoid.

609. EXERCISE. The triangle formed by joining the middle point of one of the non-parallel sides of a trapezoid with the extremities of the opposite side is equivalent to one half the trapezoid.

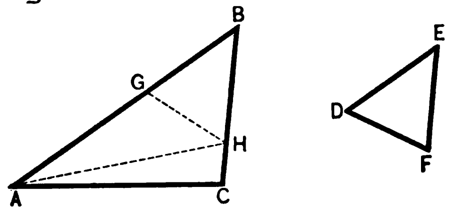
610. EXERCISE. A straight line joining the middle points of the parallel sides of a trapezoid divides it into two equivalent figures.

611. Exercise. The area of a trapezoid is 12 sq. ft. The upper and lower bases are 7 ft. and 5 ft. respectively. Find its altitude.

612. Exercise. The area of a trapezoid is 24 sq. in. The altitude is 4 in., and one of its parallel sides is 7 in. What is the other parallel side?

Proposition VIII. Theorem

613. *Triangles that have an angle in one equal to an angle in the other, are to each other as the products of the including sides.*



Let $\triangle ABC$ and DEF have $\angle B = \angle E$.

To Prove $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB \cdot BC}{DE \cdot EF}$.

Proof. Lay off $BG = ED$ and $BH = EF$. Draw GH and AH.

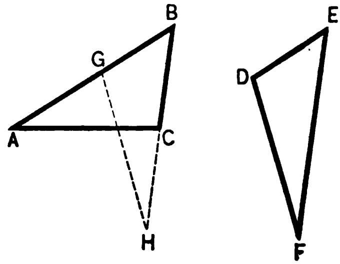
Prove $\triangle GBH \cong \triangle DEF$.

$$\frac{\triangle ABH}{\triangle BHG} = \frac{BA}{BG} \quad (?) \qquad \frac{\triangle ABC}{\triangle ABH} = \frac{BC}{BH} \quad (?)$$

$$\therefore \frac{\triangle ABC}{\triangle BHG} = \frac{AB \cdot BC}{BG \cdot BH} \text{ or } \frac{\triangle ABC}{\triangle DEF} = \frac{AB \cdot BC}{DE \cdot EF}.$$

Q.E.D.

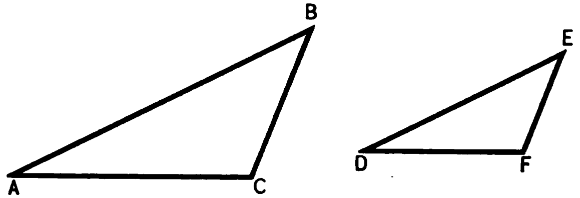
614. Exercise. Prove § 613, using this pair of triangles.



615. Exercise. The triangle ABC has $\angle B$ equal to $\angle E$ of triangle DEF. The area of ABC is double that of DEF. AB is 8 ft., BC is 6 ft., and DE is 12 ft. How long is EF?

PROPOSITION IX. THEOREM

616. *Similar triangles are to each other as the squares of their homologous sides.*



Let $\triangle ABC$ and DEF be similar.

To Prove $\dfrac{\triangle ABC}{\triangle DEF}=\dfrac{\overline{AB}^2}{\overline{DE}^2}$.

Proof. $\angle B=\angle E$. (?)

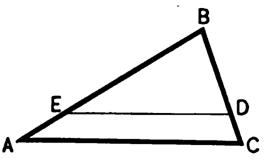
$\dfrac{\triangle ABC}{\triangle DEF}=\dfrac{AB\cdot BC}{DE\cdot EF}$. (?)

$\dfrac{AB}{DE}=\dfrac{BC}{EF}$. (?)

$\dfrac{\triangle ABC}{\triangle DEF}=\dfrac{\overline{AB}^2}{\overline{DE}^2}$. (?) Q.E.D.

617. EXERCISE. Similar triangles are to each other as the squares of their homologous altitudes.

618. EXERCISE. In the triangle ABC, ED is parallel to AC, and $CD=\frac{1}{3}DB$. How do the areas of triangles ABC and BDE compare?



619. EXERCISE. The side of an equilateral triangle is the radius of a circle. The side of another equilateral triangle is the diameter of the same circle. How do the areas of these triangles compare?

620. Exercise. Two similar triangles have homologous sides 12 ft. and 13 ft. respectively. Find the homologous side of a similar triangle equivalent to their difference.

621. Exercise. The homologous sides of two similar triangles are 3 ft. and 1 ft. respectively. How do their areas compare?

622. Exercise. Similar triangles are to each other as the squares of any two homologous medians.

623. Exercise. The base of a triangle is 32 ft., and its altitude is 20 ft. What is the area of a triangle cut off by drawing a line parallel to the base at a distance of 15 ft. from the base?

624. Exercise. A line is drawn parallel to the base of a triangle dividing the triangle into two equivalent portions. In what ratio does the line divide the other sides of the triangle?

625. Exercise. Draw a line parallel to the base of a triangle, and cutting off a triangle that shall be equivalent to one third of the remaining portion.

626. Exercise. Equilateral triangles are constructed on the sides of a right-angled triangle as bases. If one of the acute angles of the right-angled triangle is 30°, how do the largest and smallest equilateral triangles compare in area?

627. Exercise. In the triangle ABC, the altitudes to the sides AB and AC are 3 in. and 4 in. respectively. Equilateral triangles are constructed on the sides AB and AC as bases. Compare their areas.

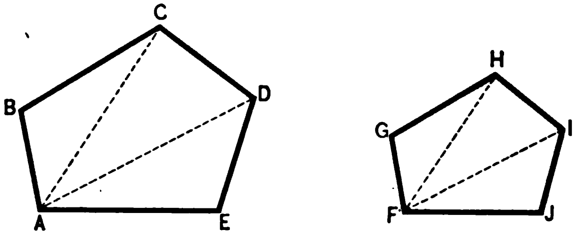
628. Exercise. The homologous altitudes of two similar triangles are 5 ft. and 12 ft. respectively. Find the homologous altitude of a triangle similar to each of them and equivalent to their sum.

629. Exercise. Draw a line parallel to the base of a triangle, and cutting off a triangle that is equivalent to $\frac{4}{5}$ of the remaining trapezoid.

630. Exercise. Through O, the point of intersection of the altitudes of the equilateral triangle ABC, lines are drawn parallel to the sides AB and BC respectively and meeting AC at x and y. Compare the areas of triangles ABC and Oxy.

PROPOSITION X. THEOREM

631. *Similar polygons are to each other as the squares of their homologous sides.*



Let $ABCDE$ and $FGHIJ$ be two similar polygons.

To Prove $\quad \dfrac{ABCDE}{FGHIJ} = \dfrac{\overline{CD}^2}{\overline{HI}^2}.$

Proof. From the vertex A draw all the possible diagonals. From F, homologous with A, draw the diagonals in $FGHIJ$.

$$\frac{\triangle ABC}{\triangle FGH} = \frac{\overline{AC}^2}{\overline{FH}^2} \quad (?)$$

$$\frac{\triangle ACD}{\triangle FHI} = \frac{\overline{AC}^2}{\overline{FH}^2} \quad (?)$$

$$\frac{\triangle ABC}{\triangle FGH} = \frac{\triangle ACD}{\triangle FHI} \quad (?)$$

Similarly prove $\quad \dfrac{\triangle ACD}{\triangle FHI} = \dfrac{\triangle ADE}{\triangle FIJ}.$

$$\frac{\triangle ABC}{\triangle FGH} = \frac{\triangle ACD}{\triangle FHI} = \frac{\triangle ADE}{\triangle FIJ} \quad (?)$$

$$\frac{\triangle ABC + \triangle ACD + \triangle ADE}{\triangle FGH + \triangle FHI + \triangle FIJ} = \frac{\triangle ACD}{\triangle FHI} (?), \text{ or } \frac{ABCDE}{FGHIJ} = \frac{\triangle ACD}{\triangle FHI}$$

$$\frac{\triangle ACD}{\triangle FHI} = \frac{\overline{CD}^2}{\overline{HI}^2} \quad (?)$$

$$\frac{ABCDE}{FGHIJ} = \frac{\overline{CD}^2}{\overline{HI}^2} \quad (?)$$

Q.E.D.

632. Corollary I. *Similar polygons are to each other as the squares of their homologous diagonals.*

633. Corollary II. *In similar polygons homologous triangles are like parts of the polygons.*

[This was shown in the proof of the proposition.]

634. Exercise. The area of a certain polygon is $2\frac{1}{4}$ times the area of a similar polygon. A side of the first is 3 ft. Find the homologous side of the second.

635. Exercise. The homologous sides of two similar polygons are 8 in. and 15 in. respectively. Find the homologous side of a similar polygon equivalent to their sum.

636. Exercise. The areas of two similar pentagons are 18 sq. yds. and 25 sq yds. respectively. A triangle of the former pentagon contains 4 sq. yds. What is the area of the homologous triangle in the second pentagon?

637. Exercise. If the triangle ADE [see figure of § 631] contains 12 sq. in., and triangle FIJ contains 9 sq. in., how do the areas of $ABCDE$ and $FGHIJ$ compare?

638. Exercise. The homologous diagonals of two similar polygons are 8 in. and 10 in. respectively. Find the homologous diagonal of a similar polygon equivalent to their difference.

639. Exercise. Connect C with m, the middle point of AD, and H with n, the middle point of FI [see figure of § 631], and prove

$$\frac{ABCDE}{FGHIJ} = \frac{\overline{Cm}^2}{\overline{Hn}^2}.$$

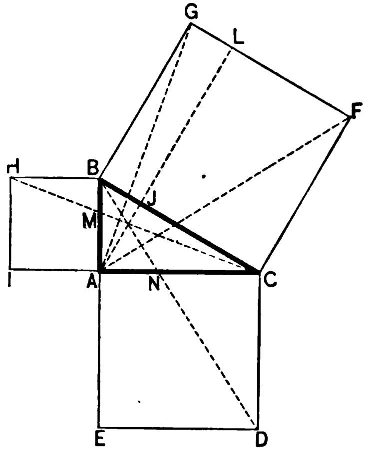
640. Exercise. If one square is double another square, what is the ratio of their sides?

641. Exercise. Construct a hexagon similar to a given hexagon and equivalent to one quarter of the given hexagon.

642. Exercise. Construct a square equivalent to $\frac{4}{9}$ of a given square.

PROPOSITION XI. THEOREM

643. *The square described on the hypotenuse of a right-angled triangle is equivalent to the sum of the squares described on the other two sides.*



Let ABC be a right-angled triangle.

To Prove $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$

Proof. Describe squares on the three sides of the triangle.
Draw $AJ \perp$ to BC, and prolong it until it meets GF at L.
Draw AF and BD.
Show that the $\triangle$s BCD and ACF are equal by § 30.
Show that the $\triangle ACF$ and the rectangle $CJLF$ have a common base and equal altitudes.

Whence, $\triangle ACF = \frac{1}{2} CJLF.$

Similarly prove $\triangle BCD = \frac{1}{2} ACDE.$

$ACDE = CJLF$ (?)

In a similar manner prove $ABHI = BGLJ.$

$$\therefore ACDE + ABHI = CJLF + BGLJ,$$

or $$\overline{AC}^2 + \overline{AB}^2 = \overline{BC}^2.$$ Q.E.D.

644. Note. The discovery of the proof of this proposition is attributed to Pythagoras (550 B.C.), and the proposition is usually called the Pythagorean Proposition.

The foregoing proof is given by Euclid (Book I., Prop. 47).

A shorter proof follows:

In the R.A. $\triangle ABC$, AJ is drawn $\perp$ to the hypotenuse.

By § 518 $$\frac{BC}{AC}=\frac{AC}{CJ}. \quad (1)$$

$$\frac{BC}{AB}=\frac{AB}{BJ}. \quad (2)$$

Whence, $$\overline{AC}^2=BC\times CJ. \quad (3)$$

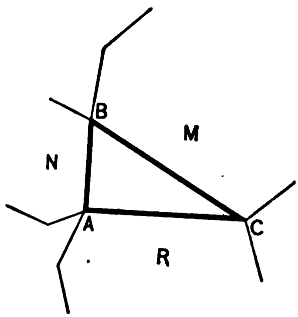
$$\overline{AB}^2=BC\times BJ. \quad (4)$$

Adding (3) and (4) $$\overline{AC}^2+\overline{AB}^2=BC\,(CJ+BJ)$$

or $$\overline{AC}^2+\overline{AB}^2=\overline{BC}^2.$$ Q.E.D.

645. Corollary I. $\overline{AC}^2=\overline{BC}^2-\overline{AB}^2$ *and* $\overline{AB}^2=\overline{BC}^2-\overline{AC}^2$, *that is, the square described on either side about the right angle is equivalent to the square described on the hypotenuse, diminished by the square described on the other side.*

646. Corollary II. *If the three sides of a right-angled triangle are made homologous sides of three similar polygons, the polygon on the hypotenuse is equivalent to the sum of the polygons on the other two sides.*



Let polygons M, N, and R be similar.

To Prove $M=N+R$.

Proof. $$\frac{N}{R}=\frac{\overline{AB}^2}{\overline{AC}^2}. \quad (?)$$

Whence $$\frac{N+R}{R}=\frac{\overline{AB}^2+\overline{AC}^2}{\overline{AC}^2}. \quad (?)$$

$$\frac{M}{R}=\frac{\overline{BC}^2}{\overline{AC}^2}. \quad (?) \quad \therefore \frac{N+R}{M}=\frac{\overline{AB}^2+\overline{AC}^2}{\overline{BC}^2}.$$

$$\overline{AB}^2+\overline{AC}^2=\overline{BC}^2. \quad \therefore N+R=M.$$ Q.E.D.

647. Corollary III. *The square described on the hypotenuse is to the square described on either of the other sides, as the hypotenuse is to the segment of the hypotenuse adjacent to that side.*

Prove $$\frac{\overline{BC}^2}{\overline{AC}^2}=\frac{BC}{JC} \text{ and } \frac{\overline{BC}^2}{\overline{AB}^2}=\frac{BC}{BJ}.$$

648. Corollary IV. *The squares described on the two sides about the right angle are to each other as the adjacent segments of the hypotenuse.*

Prove $$\frac{\overline{AB}^2}{\overline{AC}^2}=\frac{BJ}{JC}.$$

In Exercises 649–654 reference is made to the figure of § 643.

649. Exercise. Show that BI is parallel to CE.

650. Exercise. The points H, A, and D are in a straight line.

651. Exercise. AG and HC are at right angles, as are also AF and BD.

652. Exercise. If HG, FD, and IE are drawn, the three triangles HBG, FCD, and EAI are equivalent. [Use § 571.]

653. Exercise. The intercepts AM and AN are equal. [$\triangle$ BAN and CAM are similar to $\triangle$ BED and CIH respectively.]

654. Exercise. The three lines AL, BD, and HC pass through a common point.

[By means of similar triangles, show:

$$\frac{MA}{MB}=\frac{AC}{HB} \ (1) \quad \frac{NC}{AN}=\frac{CD}{AB} \ (2) \quad \text{and by Cor. IV, } \frac{BJ}{JC}=\frac{\overline{AB}^2}{\overline{AC}^2} \ (3).$$

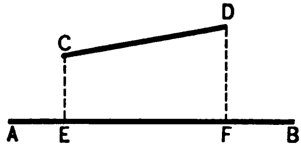
Multiply (1), (2), and (3) together, member by member.

$$\frac{MA}{MB}\times\frac{BJ}{JC}\times\frac{NC}{AN}=1. \quad \therefore AL, BD, \text{ and } HC \text{ are concurrent.}]$$

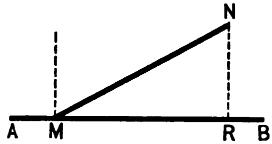
655. Exercise. The square described on the diagonal of a square is double the original square.

656. Exercise. The diagonal and side of a square are incommensurable. [See preceding exercise.]

657. Definition. The projection of CD on AB is that part of AB between the perpendiculars from the extremities of CD to AB.



EF is the projection of CD on AB.

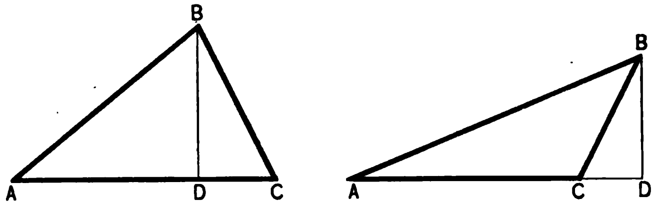


MR is the projection of MN on AB.

658. Exercise. The projection of a line upon a line parallel to it, is equal to the line itself. The projection of a line upon another line to which it is oblique is less than the line itself.

Proposition XII. Theorem

659. *In any triangle the square of a side opposite an acute angle is equivalent to the sum of the squares of the other two sides, diminished by twice the product of one of these sides and the projection of the other side upon it.*



Let ABC be a $\triangle$ in which BC lies opposite an acute angle, and AD is the projection of AB on AC.

To Prove $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 - 2\,AC \cdot AD.$

Proof. In figure (1) $DC = AC - AD.$

In figure (2) $DC = AD - AC.$

In either case $\overline{DC}^2 = \overline{AC}^2 + \overline{AD}^2 - 2\,AC \cdot AD.$

$$\overline{DC}^2 + \overline{BD}^2 = \overline{AC}^2 + \overline{AD}^2 + \overline{BD}^2 - 2\,AC \cdot AD \quad (?)$$

$$\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 - 2\,AC \cdot AD. \quad (?)$$

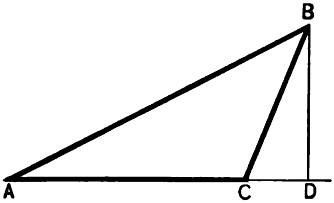
Q.E.D.

660. Exercise. Prove this proposition, using the projection of AC on AB.

661. Exercise. In a triangle ABC, $AB = 6$ ft., $AC = 5$ ft., and $BC = 7$ ft. Find the projection of AC upon BC.

Proposition XIII. Theorem

662. *In an obtuse-angled triangle the square of the side opposite the obtuse angle is equivalent to the sum of the squares of the other two sides, increased by twice the product of one of these sides and the projection of the other side upon it.*



Let ABC be an obtuse-angled △, and CD be the projection of BC on AC (prolonged).

To Prove $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2\,AC \cdot CD.$

Proof. $AD = AC + CD.$

$$\overline{AD}^2 = \overline{AC}^2 + \overline{CD}^2 + 2\,AC \cdot CD.$$

$$\overline{AD}^2 + \overline{BD}^2 = \overline{AC}^2 + \overline{CD}^2 + \overline{BD}^2 + 2\,AC \cdot CD.$$

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 + 2\,AC \cdot CD.$$

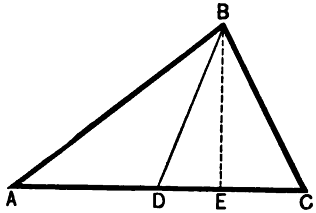
Q.E.D.

663. Corollary I. *The right-angled triangle is the only one in which the square of one side is equivalent to the sum of the squares of the other two sides.*

664. Exercise. The sides of a triangle are 6, 3, and 5. Is its greatest angle acute, obtuse, or right?

PROPOSITION XIV. THEOREM

665. *In any triangle the sum of the squares of two sides is equivalent to twice the square of one half the third side increased by twice the square of the medial line to the third side.*



Let ABC be any $\triangle$ and BD be a medial line to AC.

To Prove $\overline{AB}^2 + \overline{BC}^2 = 2\,\overline{AD}^2 + 2\,\overline{BD}^2$.

Proof. CASE I. When BD is oblique to AC.

$$\overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2 + 2\,AD \cdot DE. \quad (?)$$

$$\overline{BC}^2 = \overline{BD}^2 + \overline{DC}^2 - 2\,DC \cdot DE. \quad (?)$$

$$\overline{AB}^2 + \overline{BC}^2 = 2\,\overline{AD}^2 + 2\,\overline{BD}^2. \quad (?)$$

Q.E.D.

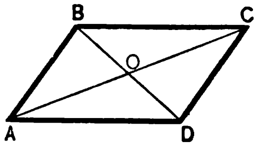
CASE II. When BD is perpendicular to AC.

$$\overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2. \quad (?) \qquad \overline{BC}^2 = \overline{DC}^2 + \overline{BD}^2. \quad (?)$$

$$\overline{AB}^2 + \overline{BC}^2 = 2\,\overline{AD}^2 + 2\,\overline{BD}^2. \quad (?)$$

Q.E.D.

666. COROLLARY I. *The sum of the squares of the sides of a parallelogram is equivalent to the sum of the squares of the diagonals.*

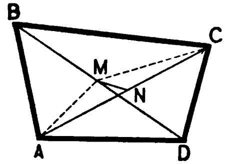


[Apply § 665 to $\triangle$ ABC and ADC and add the equations.]

667. COROLLARY II. *The sum of the squares of the sides of any quadrilateral is equivalent to the sum of the squares of the diagonals, increased by four times the square of the line joining the middle points of the diagonals.*

[To prove

$\overline{AB}^2+\overline{BC}^2+\overline{CD}^2+\overline{DA}^2=\overline{BD}^2+\overline{AC}^2+4\,\overline{MN}^2.$



Draw AM and CM.

$\overline{AB}^2+\overline{AD}^2=2\,\overline{AM}^2+2\,\overline{MD}^2.$ (?)

$\overline{BC}^2+\overline{CD}^2=2\,\overline{CM}^2+2\,\overline{MD}^2.$ (?)

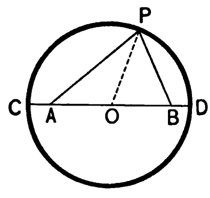
$$2(\overline{AM}^2+\overline{CM}^2)=2(2\,\overline{AN}^2+2\,\overline{MN}^2).\quad (?)$$

Add these three equations, member by member, and simplify. (Remember that $4\,\overline{MD}^2=\overline{BD}^2$.) (?)]

Show that Cor. I. is a *special case* of Cor. II.

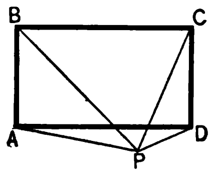
668. EXERCISE. In any triangle the difference of the squares of two sides is equivalent to the difference of the squares of their projections on the third side.

669. EXERCISE. In the diameter of a circle two points A and B are taken equally distant from the center, and joined to any point P on the circumference. Show that $\overline{AP}^2+\overline{PB}^2$ is constant for all positions of P.



670. EXERCISE. Two sides and a diagonal of a parallelogram are 7, 9, and 8 respectively. Find the length of the other diagonal.

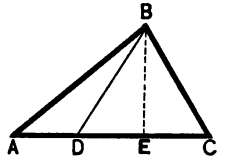
671. EXERCISE. $ABCD$ is a rectangle, and P any point from which lines are drawn to the four vertices.



Prove $\overline{AP}^2+\overline{CP}^2=\overline{BP}^2+\overline{DP}^2.$

672. Exercise. If the side AC of the triangle ABC be divided at D, so that $mAD = nDC$, and BD be drawn, prove

$$m\overline{AB}^2 + n\overline{BC}^2 = m\overline{AD}^2 + n\overline{DC}^2 + (m+n)\overline{BD}^2.$$



$$[m\overline{AB}^2 =$$
$$m(\overline{AD}^2 + \overline{BD}^2 + 2\,AD \cdot DE). \quad (?)$$
$$n\overline{BC}^2 =$$
$$n(\overline{DC}^2 + \overline{BD}^2 - 2\,DC \cdot DE). \quad (?)$$
$$m\overline{AB}^2 + n\overline{BC}^2 =$$
$$m\overline{AD}^2 + n\overline{DC}^2 + (m+n)\overline{BD}^2. \quad (?)$$

Show that § 665 is a *special case* of this exercise.]

673. Exercise. The diagonals of a parallelogram are a ft. and b ft. respectively, and one side is c ft. Find the length of the other sides.

674. Exercise. In the triangle ABC (see figure of § 672), if $AB = 9$ in., $BC = 6$ in., $AC = 10$ in., and $AD = 4$ in., find the length of BD.

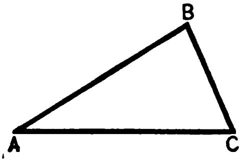
675. Exercise. Find the lengths of the medians of a triangle. [In the triangle ABC represent the lengths of the sides by a, b, and c. Show that

$$\text{Median to } AC = \tfrac{1}{2}\sqrt{2\,a^2 + 2\,c^2 - b^2}$$
$$\text{Median to } BC = \tfrac{1}{2}\sqrt{2\,b^2 + 2\,c^2 - a^2}$$
$$\text{Median to } AB = \tfrac{1}{2}\sqrt{2\,a^2 + 2\,b^2 - c^2}.]$$

676. Exercise. In the triangle ABC, the *lengths* of the sides are represented by a, b, and c (a being the length of BC opposite $\angle A$, etc.). The *sum* of the sides is called $2\,S$.

$$a + b + c = 2\,S. \quad \therefore \frac{a+b+c}{2} = S.$$

Show that
$$\frac{b+c-a}{2} = S - a,$$
$$\frac{a-b+c}{2} = S - b,$$
$$\frac{a+b-c}{2} = S - c.$$

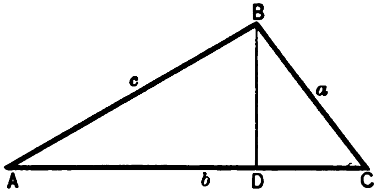


Proposition XV. Theorem

677. *The area of the triangle ABC is*

$$\sqrt{s(s-a)(s-b)(s-c)},$$

in which a, b, and c are the lengths of the three sides and 2 s their sum.



Let ABC be any $\triangle$.

To Prove $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}$.

Proof. Draw the altitude BD.

By § 659, $a^2 = b^2 + c^2 - 2\,b \cdot AD$.

Whence $AD = \dfrac{b^2 + c^2 - a^2}{2\,b}$.

In the R.A. $\triangle ABD$ by § 645,

$$\overline{BD}^2 = c^2 - \frac{(b^2+c^2-a^2)^2}{4\,b^2} = \frac{4\,b^2c^2 - (b^2+c^2-a^2)^2}{4\,b^2}$$

$$= \frac{[2\,bc - b^2 - c^2 + a^2][2\,bc + b^2 + c^2 - a^2]}{4\,b^2}$$

$$= \frac{[(a-b+c)(a+b-c)][(b+c-a)(b+c+a)]}{4\,b^2}$$

$$= \frac{4}{b^2}\left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right).$$

$\therefore \quad \overline{BD}^2 = \dfrac{4}{b^2}(s)(s-a)(s-b)(s-c).$

$BD = \dfrac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}.$ (a)

The area of $\triangle ABC = \frac{1}{2}\,b \cdot BD$.

$\therefore$ Area $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}$. Q.E.D.

678. Corollary I. *The area of an equilateral triangle is one fourth the square of a side, multiplied by* $\sqrt{3}$.

[In the formula for the area of *any* triangle, substitute a for b and also for c. Area $= \frac{1}{4} a^2\sqrt{3}$.]

679. Corollary II. *The altitude drawn to the side b in triangle ABC is* [See (*a*) of § 677.] $\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$. *Write the values of the altitudes drawn to a and c respectively.*

680. Exercise. Show that the altitude of an equilateral triangle is $\frac{1}{2}a\sqrt{3}$. (a = length of a side of the Δ.)

681. Exercise. The sides of a triangle are 5, 6, and 7. Find its area, and its three altitudes.

682. Exercise. The area of an equilateral triangle is $25\sqrt{3}$. Find its side, and also its altitude.

683. Exercise. The sides AB, BC, CD, and DA of a quadrilateral $ABCD$ are 10 in., 17 in , 13 in., and 20 in. respectively, and the diagonal AC is 21 in. What is the area of the quadrilateral?

684. Exercise. Two sides of a parallelogram are 6 in. and 7 in. respectively, and one of its diagonals is 8 in. Find its area.

685. Exercise. Two diagonals of a parallelogram are 6 in. and 8 in. respectively, and one of its sides is 5 in. Find its area, and the lengths of its altitudes.

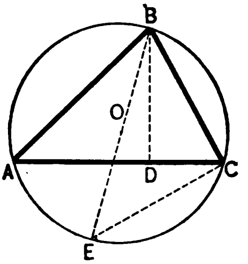
686. Exercise. The parallel sides of a trapezoid are 6 ft. and 8 ft. respectively; one of its non-parallel sides is 4 ft., and one of its diagonals is 7 ft. Find its area.

687. Exercise. The area of a triangle is 126 sq. ft., and two of its sides are 20 ft. and 21 ft. respectively. Find the third side.

[The work of this problem can be reduced by using the formula, area $=\frac{1}{4}\sqrt{4\,b^2c^2-(b^2+c^2-a^2)^2}$, and substituting 20 and 21 for b and c respectively.]

PROPOSITION XVI. THEOREM

688. *The area of a triangle is equal to the product of its three sides divided by four times the radius of the circumscribed circle.*



Let ABC be any $\triangle$ and let the lengths of its sides be represented by a, b, and c, and the radius of the circumscribed $\odot$ be called R.

To Prove $$\triangle ACB = \frac{abc}{4R}.$$

Proof. Draw the altitude BD, the diameter BE, and the chord EC.

$$\triangle ABC = \tfrac{1}{2} b \cdot BD. \quad (?) \qquad (1)$$

Prove $\triangle$s ABD and BEC mutually equiangular and similar,

whence $$\frac{BD}{AB} = \frac{BC}{BE} \text{ or } \frac{BD}{c} = \frac{a}{2R}.$$

$\therefore$ $$BD = \frac{ac}{2R}. \qquad (2)$$

Substitute (2) in (1).

$$\triangle ABC = \frac{abc}{4R}. \quad (3)$$

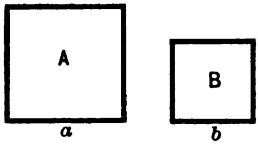
Q.E.D.

689. COROLLARY. *From the conclusion of the proposition we have* $\triangle ABC = \frac{abc}{4R}$, *whence* $R = \frac{abc}{4\triangle ABC}$. *The radius of the* *circle circumscribed about a triangle is equal to the product of the three sides divided by four times the area of the triangle.*

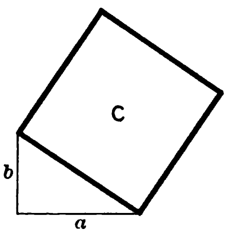
690. Exercise. The sides of a triangle are 24 ft., 18 ft., and 30 ft. respectively. Find the radius of the circumscribed circle.

Proposition XVII. Problem

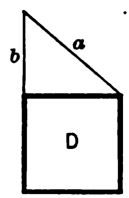
691. *To construct a square equivalent to the sum of two given squares, or equivalent to the difference of two given squares.*



Let A and B be two given squares and a and b a side of each.



Show that $C = A + B$.



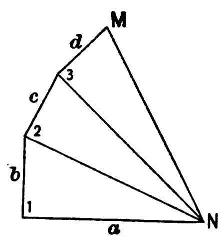
Show that $D = A - B$.

692. Corollary I. *To construct a square equivalent to the sum of several given squares.*

a, b, c, and d are the sides of the given squares.

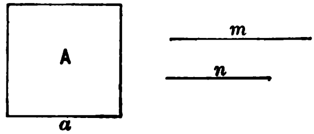
$\angle 1$, $\angle 2$, and $\angle 3$ are R.A.'s.

Show that $\overline{MN}^2 = a^2 + b^2 + c^2 + d^2$.



693. Corollary II. *Construct a square having a given ratio to a given square.*

Let A be the given square, and m and n lines having the given ratio.



[Represent a side of the required square by x.

Then $x^2 = \frac{m}{n} a^2 = \frac{ma}{n} \times a.$

Construct a line equal to $\frac{ma}{n}$ (§ 461).

Call this line c. Then $x^2 = ca$.

Find x. (§ 519.)]

694. Exercise. Construct a square equivalent to the sum or difference of a rectangle and a square.

[Construct a square equivalent to the rectangle, and then proceed as in the proposition itself.]

695. Exercise. Construct a square equivalent to the sum of the squares that have for sides 2, 4, 8, 12, and 16 units respectively.

696. Exercise. If $a = 2$ in., construct lines having the following values: $a\sqrt{2}$, $a\sqrt{3}$, $a\sqrt{5}$, $a\sqrt{6}$, $a\sqrt{7}$, and $a\sqrt{11}$.

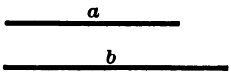
697. Exercise. If a, b, and c are given lines, construct

$$x = \frac{a^2 + 3\,bc + 4\,b^2}{2\,a + 3\,c} \quad \text{and also} \quad x = \sqrt{\frac{3\,a^2b + abc}{a + 2\,b}}.$$

698. Exercise. Construct a square whose area shall be two thirds of the area of a given square.

699. Exercise. Construct a right-angled triangle, having given the hypotenuse and the sum of the legs.

Let a be the given hypotenuse and b be the sum of the legs.



[Let x and y represent the legs.

Then
$$x + y = b,$$
$$x^2 + y^2 = a^2. \quad (§ 643.)$$

Solving these equations, we get

$$x = \tfrac{1}{2}(b + \sqrt{2\,a^2 - b^2}),$$
$$y = \tfrac{1}{2}(b - \sqrt{2\,a^2 - b^2}).$$

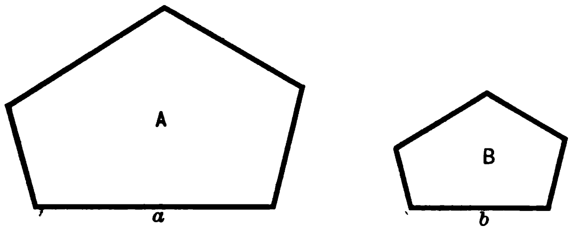
Construct these values of x and y.

Then the three sides of the triangle are known.]

700. Exercise. Construct a right-angled triangle, having given one leg, and the sum of the hypotenuse and the other leg.

Proposition XVIII. Problem

701. *To construct a polygon similar to either of two given similar polygons and equivalent to their sum.*



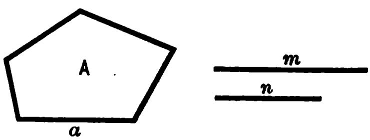
Let A and B be the two given similar polygons.

Required to construct a third polygon similar to either A or B, and equivalent to their sum.

[Construct a R.A. △ having a and b (homologous sides of A and B) for legs. On the hypotenuse of this △ construct a polygon similar to A. Show, by § 646, that this is the required polygon.]

702. Corollary I. *Construct a polygon similar to either of two given similar polygons and equivalent to their difference.*

703. Corollary II. *Construct a polygon similar to a given polygon and having a given ratio to it.*



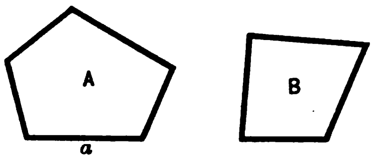
Let a be a side of the given polygon A and $\frac{m}{n}$ be the given ratio.

[Construct $x^2 = \frac{m}{n} a^2$. (§ 693.)

On a side of the square x^2 construct a polygon R similar to A.

$$\frac{R}{A} = \frac{x^2}{a^2} = \frac{\frac{m}{n} a^2}{a^2} = \frac{m}{n}. \quad (?)]$$

704. Corollary III. *Construct a polygon similar to one given polygon and equivalent to another.*



Let A and B be the two given polygons.

Required to construct a polygon similar to A and equivalent to B.

[Construct a square C equivalent to A, and a square D equivalent to B. Let c and d be sides of these squares.

Construct a line $m = \frac{ad}{c}$. (§ 461.)

On m, homologous with a, construct a polygon R similar to A

$$\frac{R}{A} = \frac{m^2}{a^2} = \frac{\frac{a^2d^2}{c^2}}{a^2} = \frac{d^2}{c^2}. \quad (?)$$

Since $A = c^2,$

$$R = d^2 = B.]$$

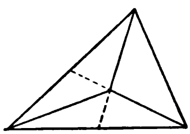
705. Exercise. Construct a quadrilateral similar to a given quadrilateral and whose area shall be 3 sq. in. (§ 704.)

706. Exercise. Construct an equilateral triangle the area of which shall be three fourths of that of a given square.

EXERCISES

1. The diagonal of a rectangle is 13 ft., one of its sides is 12 ft. What is its area?

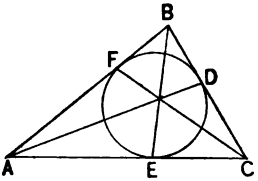
2. The square on the hypotenuse of an isosceles right-angled triangle is four times the area of the triangle.



3. The base of an isosceles triangle is 14 in., and one of the other sides is 18 in. Find the lengths of its altitudes.

4. Find a point within a triangle such that lines drawn from it to the three vertices divide the triangle into three equal parts.

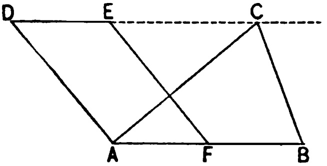
5. If a circle is inscribed in a triangle, the lines joining the points of tangency with the opposite vertices are concurrent.



$\left[\text{Show that } \frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1.\right]$

6. Given a triangle, to construct an equivalent parallelogram the perimeter of which shall equal that of the triangle.

$[FE = \frac{1}{2}(AC + BC).]$

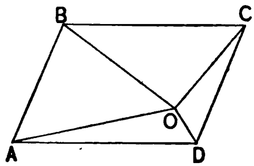


7. The sum of the three perpendiculars from a point within an equilateral triangle to the three sides is equal to the altitude of the triangle.

8. The bases of two equivalent triangles are 10 ft. and 15 ft. respectively. Find the ratio of their altitudes.

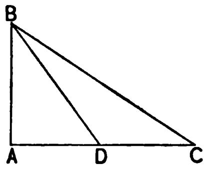
9. *ABCD* is any parallelogram, and *O* is any point within.

Prove that the sum of the areas of triangles *OAB* and *OCD* equals one half the area of the parallelogram.

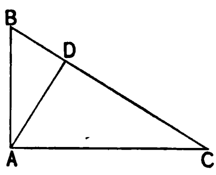


10. *ABC* is a right-angled triangle, and *BD* bisects *AC*.

Prove that $\overline{BD}^2 = \overline{BC}^2 - 3\,\overline{DC}^2$.



11. In the right-angled triangle *ABC*, *AD* is perpendicular to the hypotenuse *BC*, and the segments *BD* and *DC* are 9 ft. and 16 ft. respectively. Find the lengths of the sides, the area of the triangle, and the length of *AD*.

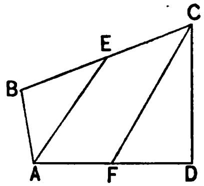


12. A square is greater than any other rectangle inscribed in the same circle.

[Show that both square and rectangle have diameters for diagonals.]

13. *ABCD* is any quadrilateral, and *AE* and *CF* are drawn to the middle points of *BC* and *AD* respectively.

Prove *AECF* equivalent to *BEA* + *CFD*.

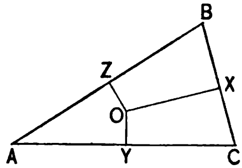


14. From any point *O* within the triangle *ABC*, *OX*, *OY*, and *OZ* are drawn perpendicular to *BC*, *CA*, and *AB* respectively.

Prove

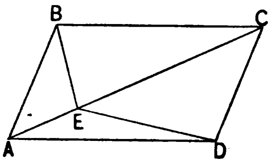
$$\overline{AZ}^2 + \overline{BX}^2 + \overline{CY}^2 = \overline{ZB}^2 + \overline{XC}^2 + \overline{YA}^2.$$

[Draw *OA*, *OB*, and *OC*. Then use § 643.]



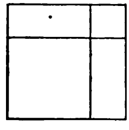
15. In the parallelogram *ABCD* any point on the diagonal *AC* is joined with the vertices *B* and *D*.

Prove triangles *ABE* and *AED* equivalent.

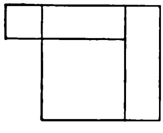


16. Draw a line through the point of intersection of the diagonals of a trapezoid dividing it into two equivalent trapezoids.

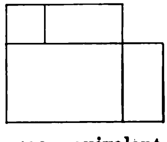
17. The square described on the sum of two lines is equivalent to the sum of the squares of the lines increased by twice their rectangle.



18. The square described on the difference of two lines is equivalent to the sum of the squares of the lines diminished by twice their rectangle.



19. The rectangle having for its sides the sum and the difference of two lines is equivalent to the difference of their squares.



20. A triangle and a rectangle having equal bases are equivalent. How do their altitudes compare?

21. Draw a straight line through a vertex of a triangle dividing it into two parts having the ratio of *m* to *n*.

22. Through a given point within or without a parallelogram draw a line dividing the parallelogram into two equivalent parts.

23. If a and b are the sides of a triangle, show that its area $= \frac{1}{4}ab$ when the included angle is 30° or 150°; $\frac{1}{4}ab\sqrt{2}$ when the included angle is 45° or 135°; $\frac{1}{4}ab\sqrt{3}$ when the included angle is 60° or 120°.

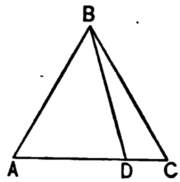
[Using either a or b for base, find the altitude of the △.]

24. If equilateral triangles are described on the three sides of a right-angled triangle, prove that the triangle on the hypotenuse is equivalent to the sum of the triangles on the other sides.

25. On a given line as a base construct a rectangle equivalent to a given rhombus.

26. Bisect a triangle by a line drawn parallel to one of its sides. [§ 616.]

27. The square of a line from the vertex of an isosceles triangle to the base is equivalent to the square of one of the equal sides diminished by the rectangle of the segments of the base [*i.e.* $\overline{BD}^2 = \overline{AB}^2 - AD \times DC$]. [Draw the altitude to AC. Use § 643.]



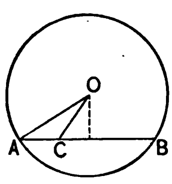
28. If, in Exercise 27, BD is drawn to a point D on the prolonged base, then $\overline{BD}^2 = \overline{AB}^2 + AD \times DC$.

29. Three times the sum of the squares on the sides of a triangle is equivalent to four times the sum of the squares on its medians. [§ 665.]

30. If the base a of a triangle is increased d inches, how much must the altitude b be diminished in order that the area of the triangle shall be unaltered.

31. OC is a line drawn from the center of the circle to any point of the chord AB.

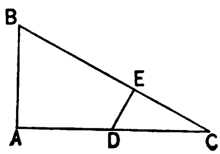
Prove that $\overline{OC}^2 = \overline{OA}^2 - AC \times CB$.



32. The lengths of the parallel sides of a trapezoid are a ft. and b ft. respectively. The two inclined sides are each c ft. Find the area of the trapezoid.

33. From the middle point D of the base of the right-angled triangle ABC, DE is drawn perpendicular to the hypotenuse BC.

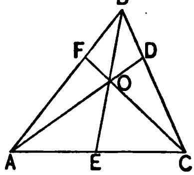
Prove that $\overline{BE}^2 - \overline{EC}^2 = \overline{AB}^2$.



34. In any circle the sum of the squares on the segments of two chords that are perpendicular to each other is equivalent to the square on the diameter. [§ 643.]

35. Construct a triangle having given its angles and its area.

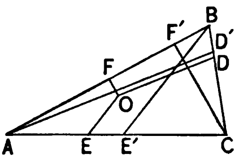
36. In the triangle ABC, AD, BE, and CF are lines drawn from the vertices and passing through a common point O.



Prove that $\frac{OE}{BE} + \frac{OD}{AD} + \frac{OF}{CF} = 1.$

$\left[\frac{OE}{BE} = \frac{\triangle AOC}{\triangle ABC}.\right.$ (?) Find similar expressions for $\frac{OD}{AD}$ and $\left.\frac{OF}{CF}.\right]$

37. From any point O within a triangle ABC, OD, OE, and OF are drawn to the three sides. From the vertices AD', BE', and CF' are drawn parallel to OD, OE, and OF respectively.

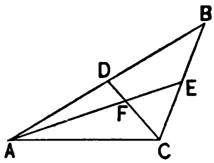


Prove that

$$\frac{OE}{BE'} + \frac{OD}{AD'} + \frac{OF}{CF'} = 1. \quad \left[\frac{OE}{BE'} = \frac{\triangle AOC}{\triangle ABC}, \text{ etc.}\right]$$

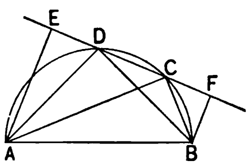
38. Given the altitude, one of the angles, and the area, construct a parallelogram.

39. The two medial lines AE and CD of the triangle ABC intersect at F. Prove the triangle AFC equivalent to the quadrilateral $BDFE$.



40. The diagonals of a trapezoid divide it into four triangles, two of which are similar, and the other two equivalent.

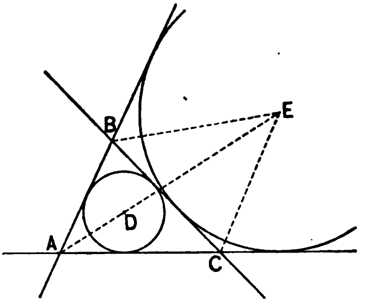
41. Any two points, C and D, in the semicircumference ACB are joined with the extremities of the diameter AB. AE and BF are drawn perpendicular to the chord DC prolonged.



Prove that $\overline{CE}^2 + \overline{CF}^2 = \overline{DE}^2 + \overline{DF}^2.$
[Use § 643.]

42. Describe four circles each of which is tangent to three lines that form a triangle.

[One of the four is the inscribed circle of the △, and its radius is denoted by r. The other three are called *escribed circles* of the triangle, and their radii are denoted by r_a, r_b, and r_c. (r_a is the radius of the escribed circle lying between the sides of ∠ A of the △.)]



43. The area of triangle $ABC = r_a\ (S - a)$.

[△ ABC = △ ABE + △ ACE − △ BEC, and r_a is the altitude of each of these △.]

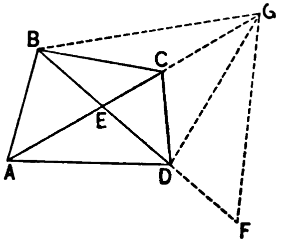
Show that $r_b\,(S-b)$ and $r_c\,(S-c)$ are also expressions for the area of triangle ABC.

44. The area of triangle $ABC = \sqrt{r \times r_a \times r_b \times r_c}$. [Ex. 43.]

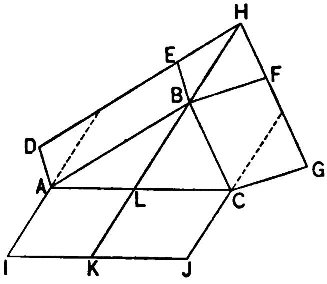
45. Prove that $r_a + r_b + r_c - r = 4\ R$ [R = radius of the circle circumscribed about △ ABC]. [Ex. 43 and § 689.]

46. Prove that $$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}.$$

47. The area of a quadrilateral is equivalent to that of a triangle having two of its sides equal to the diagonals of the quadrilateral and its included angle equal to either of the angles between the diagonals of the quadrilateral. [$DF = BE$ and $CG = AE$. Show that △ GDF = △ ABC and △ GED = △ ACD.]

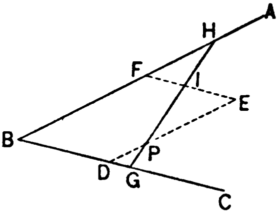


48. Parallelograms $ADEB$ and $BFGC$ are described on two sides of the triangle ABC. DE and GF are prolonged until they meet at H. HB is drawn. A third parallelogram $AIJC$ is constructed on AC, having AI equal to and parallel to BH. Prove that $AIJC$ is equivalent to the sum of $ADEB$ and $BFGC$. [$ADEB = ALKI$ and $BFGC = LCJK$.]



49. The lines joining the points of tangency of the escribed circles with the opposite vertices of the triangle ABC, are concurrent. [See Ex. 5.]

50. Deduce the Pythagorean Theorem (Prop. XI, Bk. IV) from Exercise 48.



51. Through a point P within an angle draw a line such that it and the parts of the sides that are intercepted shall contain a given area.

[Construct parallelogram $BDEF =$ required area (Ex. 38), DE passing through P. If HG is the required line, $\triangle PIE = \triangle IFH + \triangle PDG$. The $\triangle$s are similar, DP, PE, and FH are homologous sides, and DP and PE are known.]

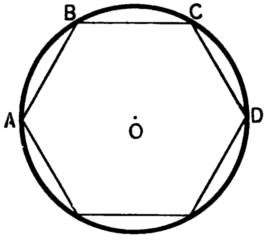
52. Is there any limit to the "given area" in Exercise 51?

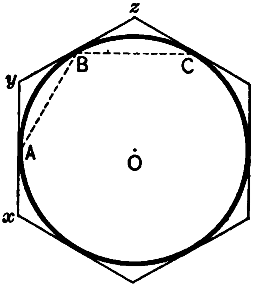
BOOK V

707. DEFINITION. A regular polygon is a polygon that is both equilateral and equiangular.

PROPOSITION I. THEOREM

708. *If the circumference of a circle is divided into three or more equal parts, the chords joining the successive points of division form a regular inscribed polygon; and tangents drawn at the points of division form a regular circumscribed polygon.*





Let the arcs AB, BC, etc., be equal.

To Prove the polygon $ABCD \cdots$ a regular inscribed polygon. [The proof is left to the student.]

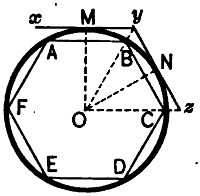
Let the arcs AB, BC, etc., be equal.

To Prove the polygon $xyz \cdots$ a regular circumscribed polygon.

Proof. [Draw the chords AB, BC, etc.

Show that the $\triangle$ AyB, BzC, etc., are isosceles $\triangle$ and are equal in all respects.]

709. COROLLARY I. *If at the middle points of the arcs subtended by the sides of a regular inscribed polygon, tangents to the circle are drawn,*



I. The circumscribed polygon formed is regular.

II. Its sides are parallel to the sides of the inscribed polygon.

III. A line connecting the center of the circle with a vertex of the outer polygon passes through a vertex of the inner polygon.

[yO bisects $\angle MON$, consequently bisects arc MN, and therefore passes through B.]

710. COROLLARY II. *If the arcs subtended by the sides of a regular inscribed polygon are bisected, and the points of division are joined with the extremities of the arcs, the polygon formed is a regular inscribed polygon of double the number of sides; and if at the extremities of the arcs and at their middle points tangents are drawn, the polygon formed is a regular circumscribed polygon of double the number of sides.*

711. COROLLARY III. *The area of a regular inscribed polygon is less than that of a regular inscribed polygon of double the number of sides; but the area of a regular circumscribed polygon is greater than that of a regular circumscribed polygon of double the number of sides.*

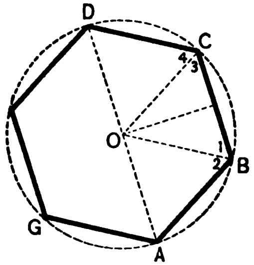
712. EXERCISE. An equiangular polygon circumscribed about a circle is regular.

713. EXERCISE. An inscribed equiangular polygon is regular if the number of its sides is odd.

714. EXERCISE. A circumscribed equilateral polygon is regular if the number of its sides is odd.

Proposition II. Theorem

715. *A circle can be circumscribed about any regular polygon; and one can also be inscribed in it.*



Let $ABC \cdots G$ be a regular polygon.

I. **To Prove** that a circle can be circumscribed about it.

Proof. Pass a circumference through three of the vertices, A, B, and C, and let O be its center.

Draw the radii OA, OB, and OC. Draw OD.

Show that $\angle 1 = \frac{1}{2} \angle B$ and $\angle 3 = \frac{1}{2} \angle C$.

Prove $\triangle$ OCB and OCD equal in all respects.

Whence $OD = OB$.

Therefore the circumference that passes through A, B, and C will also pass through D.

Similarly, it can be shown that this circumference passes through the remaining vertices. Q.E.D.

II. **To Prove** that a circle can be inscribed in the polygon.

Proof. Describe a circle about the regular polygon $AB \cdots G$.

The sides AB, BC, etc., are all equal chords of this circle, and are equally distant from the center (?).

With O as a center and this distance for a radius describe a circle.

Show that AB, BC, etc., are tangent to this circle, which is, therefore, a circle inscribed in the regular polygon. Q.E.D.

716. Definitions. The common center of the circles that are inscribed in and circumscribed about a regular polygon, is called the *center of the polygon.* The angles formed by radii drawn from this center to the vertices of the polygon are called *angles at the center.* Each angle at the center is equal to 4 right angles divided by the number of sides in the polygon. A line drawn from the center of the polygon perpendicular to a side, is an *apothem.* The apothem of a regular polygon is equal to the radius of the inscribed circle.

717. Exercise. How many degrees in the angle at the center of an equilateral triangle? Of a square? Of a regular hexagon? Of a regular polygon of n sides?

718. Exercise. How many sides has the polygon whose angle at the center is 30°? 18°?

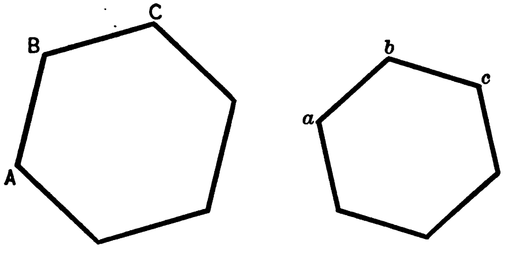
719. Exercise. In what regular polygon is the apothem one half the radius of the circumscribed circle?

720. Exercise. In what regular polygon is the apothem one half the side of the polygon?

721. Exercise. Show that an angle at the center of any regular polygon is equal to an exterior angle of the polygon.

Proposition III. Theorem

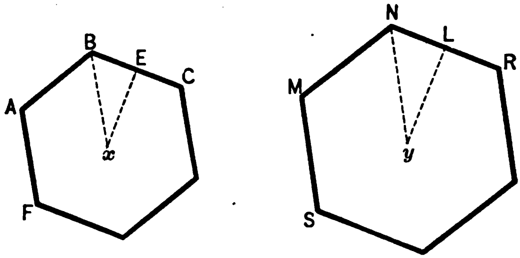
722. *Regular polygons of the same number of sides are similar.*



[Show that the polygons are mutually equiangular and have their homologous sides proportional.]

Proposition IV. Theorem

723. *The perimeters of similar regular polygons are to each other as the radii of their inscribed or of their circumscribed circles; and the polygons are to each other as the squares of the radii.*



Let $ABC \cdots F$ and $MNR \cdots S$ be two similar regular polygons.

To Prove that their perimeters are proportional to the radii of the inscribed and of the circumscribed circles, and that their areas are proportional to the squares of these radii.

Proof. Let x and y be the centers of the regular polygons. Draw xB and yN, and the apothems xE and yL.

xB and yN are the radii of the circumscribed circles and xE and yL are the radii of the inscribed circles.

$$\frac{\text{Perimeter } ABC \cdots F}{\text{Perimeter } MNR \cdots S} = \frac{BC}{NR} = \frac{Bx}{Ny} = \frac{xE}{yL}. \quad (?)$$

$$\frac{\text{Area } ABC \cdots F}{\text{Area } MNR \cdots S} = \frac{\overline{BC}^2}{\overline{NR}^2} = \frac{\overline{Bx}^2}{\overline{Ny}^2} = \frac{\overline{xE}^2}{\overline{yL}^2}. \quad (?) \qquad \text{Q.E.D.}$$

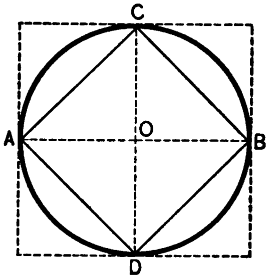
724. Exercise. Two squares are inscribed in circles, the diameters of which are 2 in. and 6 in. respectively. Compare their areas.

725. Exercise. A regular polygon, the side of which is 6 in., is circumscribed about a circle having a radius $\sqrt{3}$ in. Find the side of a similar polygon circumscribed about a circle the radius of which is 6 in.

726. Exercise. The perimeters of similar regular polygons are to each other as the diameters of their inscribed or of their circumscribed circles; and the polygons are to each other as the squares of the diameters.

Proposition V. Problem

727. *To inscribe a square in a given circle.*



Let O be the center of the given circle.

Required to inscribe a square in the circle.

Draw the diameters AB and CD at right angles.

Connect their extremities.

Prove $ACBD$ an inscribed square. (§ 708.) Q.E.F.

728. Corollary I. *Tangents to the circle at the extremities of the diameters AB and CD form a circumscribed square.*

729. Corollary II. *The side of the inscribed square is $R\sqrt{2}$.*
The side of the circumscribed square is $2R$.
The area of the inscribed square is $2R^2$.
The area of the circumscribed square is $4R^2$.

730. Corollary III. *By bisecting the arcs and drawing chords and tangents as described in* § 710, *regular polygons of* 8, 16, 32, 64, *etc., sides can be inscribed in and circumscribed about the circle.*

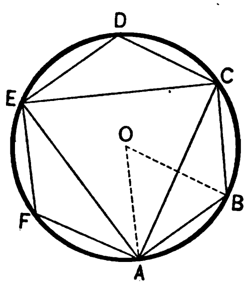
731. Exercise. The radius of a circle is 5 ft. Find the side and the area of the inscribed square.

732. Exercise. Find the side and the area of a square circumscribed about a circle, having a diameter 6 in. long.

733. Exercise. The area of a square is 16 sq. in. Find the radius of the inscribed circle and also the radius of the circumscribed circle.

Proposition VI. Problem

734. *To inscribe a regular hexagon in a circle.*



Let O be the center of the given circle.

Required to inscribe a regular hexagon in the circle.

Draw the radius OA. Lay off the chord $AB = OA$. Draw OB.

$\triangle OAB$ is equilateral, and angle O contains 60°.

$\therefore$ the arc AB is $\frac{1}{6}$ of the circumference, and the chord AB is one side of a regular hexagon.

Complete the hexagon $ABCDEF$. Q.E.F.

735. Corollary I. *The chords joining the three alternate vertices form an inscribed equilateral triangle.*

736. Corollary II. *Tangents drawn at the vertices of the inscribed hexagon and of the triangle form a regular circumscribed hexagon and a regular circumscribed triangle.*

737. Corollary III. *If the arcs are bisected and chords and tangents are drawn according to* § 710, *regular polygons of* 12, 24, 48, *etc., sides will be inscribed in and circumscribed about the circle.*

738. Exercise. The side of the inscribed equilateral triangle is $R\sqrt{3}$, and its area is $\frac{3}{4}R^2\sqrt{3}$.

739. Exercise. The side of the circumscribed equilateral triangle is $2R\sqrt{3}$, and its area is $3R^2\sqrt{3}$.

740. Exercise. The side of a regular inscribed hexagon is R, and its area is $\frac{3}{2}R^2\sqrt{3}$.

741. Exercise. The side of a regular circumscribed hexagon is $\frac{2}{3}R\sqrt{3}$, and its area is $2R^2\sqrt{3}$.

742. Exercise. The area of a regular inscribed hexagon is double that of an equilateral triangle inscribed in the same circle. [Show this in two ways: 1st, by comparing the values of their areas as derived in §§ 738 and 740; 2d, by a geometrical demonstration using the figure of § 734.]

743. Exercise. What is the area of a regular hexagon inscribed in a circle, the radius of which is 4 in.?

744. Exercise. The area of a regular inscribed hexagon is 10 sq. in. What is the area of a regular hexagon circumscribed about the same circle?

745. Exercise. The area of an equilateral triangle is $48\sqrt{3}$ sq. ft. Find the radii of the inscribed and of the circumscribed circles.

746. Exercise. The area of a regular hexagon is $54a^2\sqrt{3}$. Find the radii of the inscribed and of the circumscribed circles.

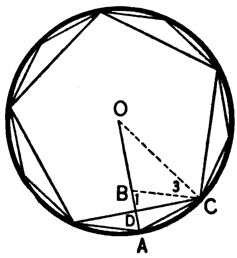
747. Exercise. Show that the circumscribed equilateral triangle is 4 times the inscribed equilateral triangle; that the circumscribed square is 2 times the inscribed square; and that the circumscribed regular hexagon is $\frac{4}{3}$ of the inscribed regular hexagon.

748. Exercise. Divide a circumference into quadrants by the use of compasses only.

[Suggestion. The side of an inscribed square is the altitude of an isosceles triangle whose base is $2R$ and one of whose sides is $R\sqrt{3}$.]

Proposition VII. Problem

749. *To inscribe a regular decagon in a circle.*



Let O be the center of the given circle.

Required to inscribe a regular decagon in the circle.

Draw the radius OA. Divide it in extreme and mean ratio, OB being the greater segment.

Lay off $AC = OB$. Draw BC and OC.

By definition (Art. 551), $\frac{OA}{OB} = \frac{OB}{BA}$.

$$\frac{OA}{AC} = \frac{AC}{BA}. \quad (?)$$

$\triangle$s OAC and BAC are similar. (§ 495.)

$\therefore$ $\triangle BAC$ is isosceles, and $AC = BC$. (?)

$\triangle BOC$ is isosceles. (?)

$$\angle 1 = \angle 3 + \angle O \ (?) \text{ or } \angle 1 = 2\angle O. \quad (?)$$

$$\angle A = 2\angle O \ (?) \text{ and } \angle ACO = 2\angle O. \quad (?)$$

$$\angle A + \angle ACO + \angle O = 180°. \quad (?)$$

$$2\angle O + 2\angle O + \angle O = 180°. \quad (?) \quad \therefore \angle O = 36°.$$

$\therefore$ the arc AC, the measure of $\angle O$, contains 36° of arc, and is $\frac{1}{10}$ of the circumference.

The circumference can therefore be divided into ten parts, each equal to the arc AC, and the chords joining the points of division form a regular inscribed decagon. Q.E.F.

750. COROLLARY I. *The chords joining the alternate vertices of a regular inscribed decagon form a regular inscribed pentagon.*

751. COROLLARY II. *Tangents drawn at the vertices of the regular inscribed pentagon and decagon form a regular circumscribed pentagon and a regular circumscribed decagon.*

752. COROLLARY III. *If the arcs are bisected and chords and tangents are drawn according to* § 710, *regular inscribed and circumscribed polygons of* 20, 40, 80, *etc., sides will be formed.*

753. EXERCISE. The length of the side of a regular inscribed decagon is $\frac{1}{2}(\sqrt{5}-1)r$.

754. EXERCISE. Find the length of a side of a regular inscribed pentagon. [In the R.A. $\triangle ADC$ (see the figure of § 749), AC is the side of the decagon, and AD is one half the difference between the radius and the side of the decagon.]

Ans. $\dfrac{\sqrt{10-2\sqrt{5}}}{2}r$.

755. EXERCISE. Show that the sum of the squares described on the sides of a regular inscribed decagon and of a regular inscribed hexagon equals the square described on the side of a regular inscribed pentagon.

[Represent the sides of the pentagon, hexagon, and decagon by p, h, and d, respectively.

In the figure of § 749,

$$\overline{DC}^2 = \overline{AC}^2 - \overline{AD}^2,$$

or

$$(\tfrac{1}{2}p)^2 = d^2 - \left(\frac{h-d}{2}\right)^2,$$

whence

$$p^2 = 3d^2 - h^2 + 2hd. \tag{1}$$

By § 551 $\quad \dfrac{h}{d} = \dfrac{d}{h-d}, \quad$ whence $hd = h^2 - d^2$. (2)

From (1) and (2) $\quad p^2 = d^2 + h^2$.

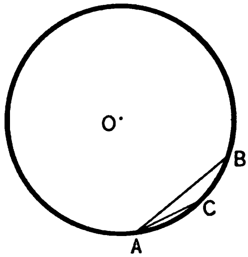
Give also an algebraic proof.]

756. EXERCISE. What is the length of the side of a regular decagon inscribed in a circle having a diameter 4 in. long?

757. EXERCISE. If the side of a regular pentagon is $2\sqrt{5}$ in., show that the radius of the circumscribed circle is $\sqrt{10+2\sqrt{5}}$ in.

PROPOSITION VIII. PROBLEM

758. *To inscribe a regular pentedecagon in a circle.*



Let O be the center of the given circle.

Required to inscribe a regular polygon of fifteen sides in the circle.

Lay off the chord AB = side of regular inscribed hexagon, and the chord AC = side of regular inscribed decagon.

The arc AB contains 60°, (?) and the arc AC, 36°. (?)

$\therefore$ the arc BC contains 24° and is $\frac{1}{15}$ of the circumference. The circumference can therefore be divided into fifteen parts, each equal to BC; and the chords joining the points of division form a regular inscribed pentedecagon. Q.E.F.

759. COROLLARY I. *Tangents drawn at the vertices of the inscribed pentedecagon form a regular circumscribed pentedecagon.*

760. COROLLARY II. *If the arcs are bisected, and chords and tangents are drawn as described in* § 710, *regular inscribed and circumscribed polygons of* 30, 60, 120, *etc., sides will be formed.*

761. SCHOLIUM. In Propositions V., VI., VII., and VIII. we have seen that the circumference can be divided into the following numbers of equal parts:

2,	4,	8,	16	...	2^n	n being any positive integer.
3,	6,	12,	24	...	3×2^n	
5,	10,	20,	40	...	5×2^n	
15,	30,	60,	120	...	15×2^n	

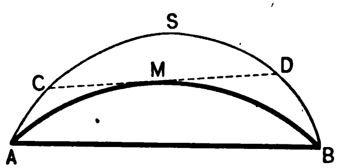
The mathematician Gauss has shown that it is possible to divide the circumference into $2^n + 1$ equal parts, n being a positive integer and $2^n + 1$ a prime number.

It is therefore possible, by the use of ruler and compasses, to divide the circumference into 2, 3, 5, 17, 257, etc., equal parts.

[An elementary explanation of the division of the circumference into seventeen equal parts is given in Felix Klein's "Vorträge über ausgewählte Fragen der Elementar Geometrie."]

Proposition IX. Theorem

762. *The arc of a circle is less than any line that envelops it and has the same extremities.*



Let *AMB* be the arc of circle and *ASB* any other line enveloping it and passing through *A* and *B*.

To Prove $AMB < ASB$.

Proof. Of all the lines (*AMB*, *ASB*, etc.) that can be drawn through *A* and *B*, and including the segment or *area AMB*, there must be one of minimum length.

ASB cannot be the minimum line, for draw the tangent *CD* to the arc *AMB*.

$$CD < CSD. \quad (?)$$

$$ACDB < ASB. \quad (?)$$

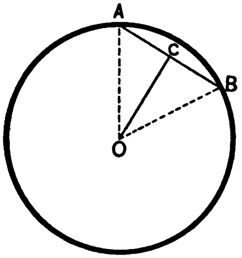
The same can be shown of every other line (except *AMB*) passing through *A* and *B* and including the area *AMB*.

∴ the arc *AMB* is the minimum line. Q.E.D.

763. Corollary I. *The circumference of a circle is less than the perimeter of a circumscribed polygon and greater than the perimeter of an inscribed polygon.*

Proposition X. Theorem

764. *If the number of sides of a regular inscribed polygon is indefinitely increased, its apothem approaches the radius as a limit.*



Let AB be the side of a regular inscribed polygon and OC be its apothem.

To Prove that OC approaches the radius as its limit when the number of sides is indefinitely increased.

Proof. $OA > OC$. (?)

$OA - OC < AC$. (?) $\therefore OA - OC < AB$.

By increasing the number of sides AB can be made as small as we please, but not equal to zero. AB consequently approaches zero as a limit, and since $OA - OC < AB$, $OA - OC$ approaches zero as its limit; and OC approaches OA as its limit. Q.E.D.

765. Corollary. If the number of sides of a regular circumscribed polygon is indefinitely increased, the distance from a vertex to the center of the circle approaches the radius as a limit.

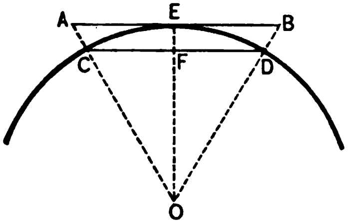
[Proof similar to § 764.]

Proposition XI. Theorem

766. *If a regular polygon is inscribed in or circumscribed about a circle and the number of its sides is indefinitely increased,*

I. *The perimeter of the polygon approaches the circumference as its limit.*

II. *The area of the polygon approaches the area of the circle as its limit.*



Let AB be the side of a regular circumscribed polygon, and CD (parallel to AB) be the side of a similar inscribed polygon.

I. **To Prove** that the perimeters of the polygons approach the circumference of the circle as a limit when the number of sides is indefinitely increased.

Proof. Draw OA, OB, and OE.

OA passes through C and OB through D. (?)

Let P and p stand for the perimeters of the circumscribed and inscribed polygons respectively.

$$\frac{P}{p} = \frac{OE}{OF}. \qquad (?)$$

$$\frac{P - p}{P} = \frac{OE - OF}{OE} \qquad (?)$$

or

$$P - p = \frac{P}{OE}(OE - OF).$$

As shown in the preceding proposition, $OE - OF$ can be made as small as we please, though not equal to zero; and since $\frac{P}{OE}$ does not increase, $\frac{P}{OE}(OE - OF)$, or its equal $P - p$, can be decreased at pleasure.

Since P is always greater than the circumference, and p is always less than the circumference, the difference between the circumference and either perimeter is less than the difference $P - p$, and can consequently be made as small as we please, but not equal to zero.

The circumference is therefore the common limit of the two perimeters as the number of sides is indefinitely increased.

Q.E.D.

II. **To Prove** that the areas of the polygons approach the area of the circle as a limit, when the number of sides is indefinitely increased.

Proof. Let S and s stand for the areas of the circumscribed and inscribed polygons respectively.

$$\frac{S}{s} = \frac{\overline{OE}^2}{\overline{OF}^2}. \quad (?)$$

$$\frac{S - s}{S} = \frac{\overline{OE}^2 - \overline{OF}^2}{\overline{OE}^2} = \frac{\overline{CF}^2}{\overline{OE}^2}. \quad (?)$$

$$S - s = \frac{S}{\overline{OE}^2}(\overline{CF}^2)$$

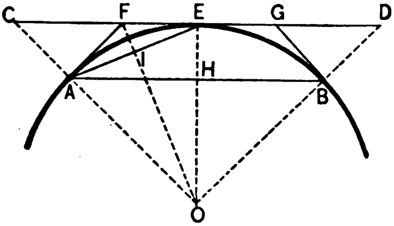
As the number of sides is indefinitely increased, CD approaches zero as a limit, as does also CF, and consequently $\overline{CF}^2$.

[The remainder of the proof is similar to that of Case I. of this proposition.] Q.E.D.

767. Exercise. If, as is shown in § 766, the difference between P and p can be made as small as we please, why is not p the limit of P? (See definition of limit.)

Proposition XII. Problem

768. *Given the perimeters of a regular inscribed polygon and of a similar circumscribed polygon, to find the perimeters of regular inscribed and circumscribed polygons of double the number of sides.*



Let AB be a side of a regular inscribed polygon of n sides,
CD (parallel to AB) a side of a regular circumscribed polygon of n sides,
AE a side of a regular inscribed polygon of $2n$ sides,
FG a side of a regular circumscribed polygon of $2n$ sides.

Required to find the perimeters of the inscribed and circumscribed polygons of $2n$ sides.

Call the perimeter of the inscribed polygon of n sides p,
the perimeter of the circumscribed polygon of n sides P,
the perimeter of the inscribed polygon of $2n$ sides p',
the perimeter of the circumscribed polygon of $2n$ sides P'.

Then $AB = \frac{p}{n}$ and $AH = \frac{p}{2n}$. $CD = \frac{P}{n}$ and $CE = \frac{P}{2n}$.

$$AE = \frac{p'}{2n}. \quad FG = \frac{P'}{2n}.$$

$$\frac{P}{p} = \frac{OC}{OE} \quad (?) = \frac{CF}{FE}. \quad (\S\ 502.)$$

$$\frac{P+p}{2p} = \frac{CF+FE}{2\,FE} = \frac{CE}{FG} = \frac{P}{P'}.$$

$$\therefore P' = \frac{2p \times P}{P+p}. \quad \text{(I.)}$$

Prove $\triangle$ *IFE* and *AEH* similar,

whence
$$\frac{AH}{AE}=\frac{IE}{FE}.$$

$$\frac{AH}{AE}=\frac{p}{p'} \text{ and } \frac{IE}{FE}=\frac{p'}{P'}. \quad (?)$$

$$\therefore \frac{p}{p'}=\frac{p'}{P'} \text{ and } p'=\sqrt{p \times P'}. \quad \text{(II.)}$$

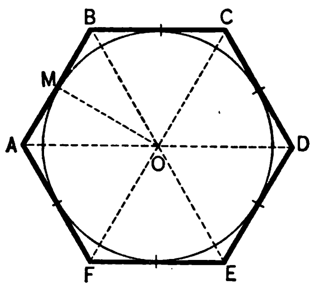
Since p and P are given, Formula I. gives the value of P'; then from Formula II. the value of p' can be derived. Q.E.F.

769. EXERCISE. The side of an inscribed square is $3\sqrt{2}$ and the side of a circumscribed square is 6. Find the sides of regular octagons inscribed in and circumscribed about the same circle.

770. EXERCISE. Find the perimeters of regular dodecagons (12-sided polygons) inscribed in and circumscribed about a circle having a diameter 12 in. long.

PROPOSITION XIII. THEOREM

771. *The area of a regular polygon is equal to one half the product of its perimeter and apothem.*



Let *ABCDEF* be a regular polygon.

To Prove that its area is equivalent to one half the product of its perimeter and apothem.

Suggestion. The altitude of each $\triangle$ is the apothem, and the polygon is equivalent to the sum of the triangles.

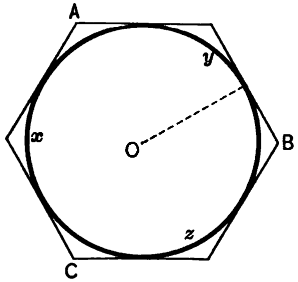
772. Corollary. *The area of any circumscribed polygon is equal to one half the product of its perimeter and the radius of its inscribed circle.*

773. Exercise. The perimeter of a polygon circumscribed about a circle having a 5 ft. radius, is 32 ft. What is its area?

774. Exercise. The side of a regular hexagon is 6 in. Find its area. [*Suggestion.* First find its apothem.]

Proposition XIV. Theorem

775. *The area of a circle is equal to one half the product of its circumference and radius.*



Let xyz be any circle.

To Prove area $xyz = \frac{1}{2}$ circumference $\times$ radius.

Proof. Circumscribe a regular polygon ABC about the circle xyz.

Area $ABC = \frac{1}{2}$ perimeter $\times$ apothem. (?)

If the number of sides of the polygon is increased, the area changes as does also the perimeter, and yet the area is *always* equal to $\frac{1}{2}$ perimeter $\times$ apothem. So the two members of the above equation may be regarded as two variables that are always equal. Since each is approaching a limit, their limits must be equal. [§ 341.]

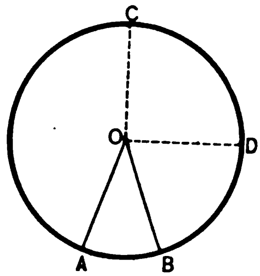
The limit of area ABC = area of circle. (?)

The limit of the perimeter = circumference. (?)

The apothem is constant and equals the radius.

$\therefore$ area $xyz = \frac{1}{2}$ circumference $\times$ radius. Q.E.D.

776. COROLLARY. *The area of a sector is equal to one half the product of its arc and radius.*



To Prove area $AOB = \frac{1}{2} AB \times R$.

Proof. Construct the quadrant COD.

$$\frac{\text{sector } AOB}{\text{sector } COD} = \frac{AB}{CD} \quad (\S\ 345.)$$

$$\frac{\text{sector } AOB}{4 \text{ sector } COD} = \frac{AB}{4\ CD}$$

or
$$\frac{\text{sector } AOB}{\text{circle}} = \frac{AB}{\text{circumf.}}.$$

$$\frac{\text{sector } AOB}{\frac{1}{2} \text{ circumf.} \times R} = \frac{AB}{\text{circumf.}} \quad \therefore \text{ sector } AOB = \tfrac{1}{2} AB \times R.$$

777. EXERCISE. The radius of a circle is 100 ft. and its circumference is 628.32 ft. Find its area.

778. EXERCISE. The area of a sector is 68 sq. in., and its radius is 8 in. How long is its arc?

779. EXERCISE. The area of a circle is 100 sq. ft. The area of a sector of this circle is $12\frac{1}{2}$ sq. ft. How many degrees in the arc of the sector?

780. EXERCISE. The area of a circle is 10 sq. ft. Find the area of a segment whose arc contains 60°.

Suggestion. Find the area of the sector having arc = 60°. Subtract the area of the triangle formed by the chord and the radii from the area of the sector.

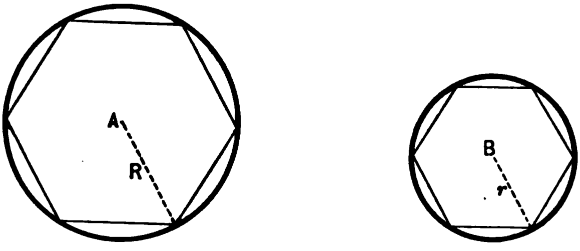
781. EXERCISE. The circumference of a circle is 94.248 ft. The side of an inscribed equilateral triangle is $15\sqrt{3}$ ft. Find the area of the circle.

782. EXERCISE. The area of a circle is 314.16 sq. in. The perimeter of a regular inscribed hexagon is 60 in. Find the circumference of the circle.

783. EXERCISE. Find the area of the part of the circle of § 782 lying between its circumference and the perimeter of a regular hexagon inscribed in the circle.

PROPOSITION XV. THEOREM

784. ***The circumferences of two circles are to each other as their radii, and the circles are to each other as the squares of their radii.***



Let A and B be two circles and R and r be their radii.

To Prove $$\frac{\text{circumf. } A}{\text{circumf. } B}=\frac{R}{r}.$$

Proof. Inscribe similar regular polygons in the two circles. Let P and p denote the perimeters of these polygons.

$$\frac{P}{p}=\frac{R}{r} \quad (?) \quad \text{or} \quad \frac{P}{R}=\frac{p}{r} \quad (1)$$

As the number of sides is indefinitely increased, P and p approach circumference A and circumference B respectively as their limits. (?)

The members of equation (1) may therefore be regarded as two variables that are always equal, and since each is approaching a limit, their limits are equal. (?)

$$\therefore \quad \frac{\text{circumf. } A}{R}=\frac{\text{Circumf. } B}{r}$$

or $$\frac{\text{circumf. } A}{\text{circumf. } B}=\frac{R}{r}.$$

Similarly, show that $\frac{\text{circle } A}{\text{circle } B}=\frac{R^2}{r^2}$. Q.E.D.

785. Corollary I. *The circumferences of two circles are to each other as their diameters, and the circles are to each other as the squares of their diameters.*

786. Corollary II. *The ratio of the circumference of a circle to its diameter is constant; that is, it is the same for all circles.*

By § 785,
$$\frac{\text{circumf. } A}{\text{circumf. } B} = \frac{\text{diam. } A}{\text{diam. } B},$$

or
$$\frac{\text{circumf. } A}{\text{diam. } A} = \frac{\text{circumf. } B}{\text{diam. } B}.$$

The value of this constant is denoted by the Greek letter π.

Thus,
$$\frac{\text{circumf. } A}{\text{diam. } A} = \pi.$$

Whence
$$\text{circumf. } A = \pi \text{ diam. } A.$$

i.e. *The circumference of a circle is π times its diameter.*

If, in the formula for the area of a circle,

$$\text{area} = \tfrac{1}{2} \text{ circumf.} \times R,$$

the value of the circumference just derived is substituted, we obtain

$$\text{area} = \pi R^2.$$

i.e. *The area of a circle is π times the square of its radius.*

787. Definition. *Similar arcs* are arcs that subtend equal angles at the center.

Since the intercepted arcs are the measures of the angles at the center, similar arcs contain the same number of degrees of arc, and are consequently like parts of their circumferences.

Similar sectors are sectors the radii of which include equal angles, or intercept similar arcs.

Similar segments are segments whose arcs are similar.

788. Corollary III. *Similar arcs are to each other as their radii.* [See definition.]

789. Corollary IV. *Similar sectors are to each other as the squares of their radii.* [§§ 776 and 788.]

790. Corollary V. *Similar segments are to each other as the squares of their radii.*

791. **Exercise.** **The circumferences of two circles are 942.48 ft. and 157.08 ft. respectively.**

The diameter of the first is 300 ft. Find the diameter of the second.

792. **Exercise.** **What is the ratio of the areas of the two circles of the preceding exercise?**

793. **Exercise.** **How many units in the radius of a circle, the area and circumference of which can be expressed by the same number?**

Proposition XVI. Problem

794. *To find an approximate value of* π.

The perimeter of a circumscribed square (see § 729) is $4\,D$ ($D =$ diameter).

The perimeter of an inscribed square is $2\sqrt{2}\,D = 2.8284271\,D$.

Substituting $4\,D$ for P and $2.8284271\,D$ for p in the formulas $P' = \frac{2p \times P}{P + p}$ (1) and $p' = \sqrt{p \times P'}$ (2), we get P' or the perimeter of the circumscribed octagon $= 3.3137085\,D$, and p' or the perimeter of the inscribed octagon $= 3.0614675\,D$.

Substituting $3.3137085\,D$ for P and $3.0614675\,D$ for p in formulas (1) and (2), we obtain values for the perimeters of the circumscribed and the inscribed polygons of sixteen sides.

Substituting these values, the perimeters of polygons of thirty-two sides are obtained.

Continuing in this way, the following table is formed:

NUMBER OF SIDES.	PERIMETER OF CIRCUMSCRIBED POLYGON.	PERIMETER OF INSCRIBED POLYGON.
4	4.0000000 *D*	2.8284271 *D*
8	3.3137085 *D*	3.0614675 *D*
16	3.1825979 *D*	3.1214452 *D*
32	3.1517249 *D*	3.1365485 *D*
64	3.1441184 *D*	3.1403312 *D*
128	3.1422236 *D*	3.1412773 *D*
256	3.1417504 *D*	3.1415138 *D*
512	3.1416321 *D*	3.1415729 *D*
1024	3.1416025 *D*	3.1415877 *D*
2048	3.1415951 *D*	3.1415914 *D*
4096	3.1415933 *D*	3.1415923 *D*
8192	3.1415928 *D*	3.1415926 *D*

The circumference of the circle therefore lies between 3.1415926 *D* and 3.1415928 *D*.

For ordinary accuracy the value of π is taken as 3.1416.

NOTE. — The value of π has been carried out over seven hundred decimal places. [See article on "Squaring the Circle" in the Encyclopædia Britannica.]

The value of π to thirty-five decimal places is

3.14159265358979323846264338327950288.

By higher mathematics, the diameter and circumference of the circle have been shown to be incommensurable, so no *exact* expression for their ratio can be obtained.

795. EXERCISE. The radius of a circle is 10 in. Find its circumference and its area.

796. EXERCISE. The area of a circle is 7854 sq. ft.. Find its circumference.

797. Exercise. The circumference of a circle is 50 in. What is its area?

798. Exercise. The radius of a circle is 50 ft. What is the area of a sector whose arc contains 40°?

799. Exercise. The radius of a circle is 10 ft. The area of a sector of that circle is 120 sq. ft. What is its arc in degrees?

EXERCISES

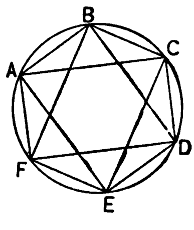
1. In a regular polygon of n sides, diagonals are drawn from one vertex. What angles do they make with each other?

2. Show that the altitude of an inscribed equilateral triangle is $\frac{3}{4}$ of the diameter, and that the altitude of a circumscribed equilateral triangle is 3 times the radius.

3. The radii of two circles are 4 in. and 6 in. respectively. How do their areas compare?

4. Find the area of the ring between the circumferences of two concentric circles the radii of which are a and b respectively.

5. The area of a regular inscribed hexagon is a mean proportional between the areas of the inscribed and the circumscribed equilateral triangles. [See Ex. to Prop. 6.]



6. The diagonals joining the alternate vertices of a regular hexagon form by their intersection a regular hexagon having an area one third of that of the original hexagon.

7. Find the area of the six-pointed star in the figure of Exercise 6 in terms of the radius of the circle.

8. From any point within a regular polygon of n sides, perpendiculars are drawn to the sides. Prove that the sum of these perpendiculars is equal to n times the apothem of the polygon.

[Join the point with the vertices and obtain an expression for the area of the polygon. Compare this with the expression for the area obtained from § 771.]

9. Construct a circle that shall be double a given circle (§ 784).

10. Construct a circle that shall be one half a given circle.

11. Construct a circle equivalent to the sum of two given circles; also one equivalent to their difference. [§ 646.]

12. If two circles are concentric, show that the area of the ring between their circumferences is equal to the area of a circle having for its diameter a chord of the larger circle that is tangent to the smaller.

13. Find the area of the sector of a circle intercepting an arc of 50°, the radius of the circle being 10 ft. [§ 776.]

14. The radius of a circle is 20 ft. What is the angle of a sector having an area of 300 sq. ft.?

15. The radius of a circle is 20 ft., and the area of a sector of the circle is 300 sq. ft. Find the area of a similar sector in a circle having a radius 50 ft. long.

16. What is the radius of a circle having an area equal to 16 times the area of a circle with a radius 5 ft. long?

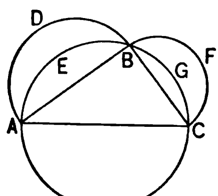
17. Find the area of a circle circumscribed about a square having an area of 600 sq. ft. [§ 729.]

18. Show that the area of a circumscribed equilateral triangle is greater than that of a square circumscribed about the same circle.

19. Four circles, each with a radius 5 ft. long, have their centers at the vertices of a square, and are tangent. Find the area of a circle tangent to all of them.

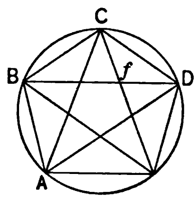
20. How many degrees in the arc, the length of which is equal to the radius of the circle?

21. A circle is circumscribed about the right-angled triangle ABC. Semicircles are described on the two legs as diameters. Prove that the sum of the crescents $ADBE$ and $BFCG$ is equivalent to the triangle ABC.



22. The radius of a regular inscribed polygon is a mean proportional between its apothem and the radius of a similar circumscribed polygon.

23. If the bisectors of the angles of a polygon meet in a point, a circle can be inscribed in the polygon.



24. The diagonals of a regular pentagon form by their intersection a second regular pentagon.

25. Any two diagonals of a regular pentagon not drawn from a common vertex divide each other into extreme and mean ratio. [$\triangle$s ABC and CfD are similar.]

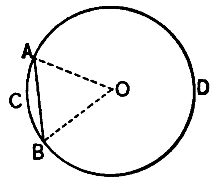
26. Divide an angle of an equilateral triangle into five equal parts.

27. If two angles at the centers of unequal circles are subtended by arcs of equal *length*, the angles are inversely proportional to the radii of the circles.

28. The apothem of a regular inscribed pentagon is equal to one half the sum of the radius of the circle and a side of a regular inscribed decagon.

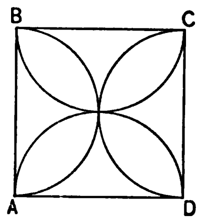
29. If two chords of a given circle intersect each other at right angles, and on the four segments of the chords as diameters, circles are described, the sum of the four circles is equivalent to the given circle. [Ex. 34, page 217.]

30. Divide a circle into three equivalent parts by concentric circles (§ 784).



31. The radius of a given circle ABD is 10 ft. Find the areas of the two segments BCA and BDA into which the circle is divided by a chord AB equal in length to the radius. [Subtract area of △ from area of sector.]

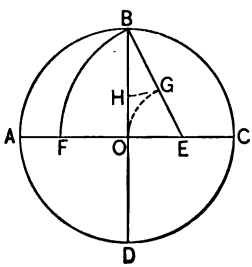
32. Find the radius of a circle that is doubled in area by increasing its radius one foot.



33. On the sides of a square as diameters, four semicircles are described within the square, forming four leaves. If the side of the square is a, find the area of the leaves.

34. In a given equilateral triangle inscribe three equal circles tangent to each other and to the sides of the triangle.

35. In a given circle inscribe three equal circles tangent to each other and to the given circle.



36. In the circle $ABCD$, the diameters AC and BD are at right angles to each other. With E, the middle point of OC, as a center, and EB as a radius, the arc BF is described. Prove that the radius OA is divided into extreme and mean ratio at F.

[Describe arc OG with E as center, and arc GH with B as center.]

37. If a regular polygon of n sides be circumscribed about a circle, the sum of the perpendiculars from the points of contact to any tangent to the circle is equal to n times the radius.

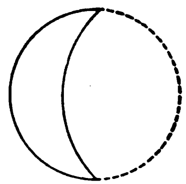
[If A, B, C, D, etc., are the points of contact of the polygon and P the point at which the tangent is drawn, the sum of the ⊥s from A, B, etc., on tangent at $P =$ sum of ⊥s from P to tangents drawn at A, B, etc.; and this by Ex. 8 $= nR$.]

38. The sum of the perpendiculars from the vertices of a regular inscribed polygon to any line without the circle is equal to n times the perpendicular from the center of the circle to the line.

[Draw a tangent to the ⊙ parallel to the given line, and then use Ex. 39.]

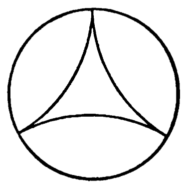
39. The sum of the squares of the lines drawn from any point in the circumference to the vertices of a regular inscribed polygon is equal to $2\,nR^2$.

[Using notation of Ex. 39, show that the square of the line from the given point P to each vertex $= 2\,R$ times the ⊥ from the vertex to a tangent at P. Add these equations and use Ex. 39.]



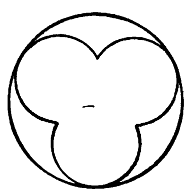
40. A crescent-shaped region is bounded by a semi-circumference of radius a, and another circular arc whose center lies on the semi-circumference produced. Find the area and the perimeter of the region.

[Show that the arc is a quadrant in a ⊙ with radius $= a\sqrt{2}$.]



41. Three points divide a circumference into equal parts. Through each pair of these points an arc of a circle is described tangent to the radii drawn to the points and lying wholly within the circle. Find the perimeter of the figure thus formed, and show that its area is $3\,(\sqrt{3} - \frac{1}{2}\pi)\,a^2$, where a denotes the radius of the circle.

[Show that each arc is $\frac{1}{6}$ of a circumference with radius $a\sqrt{3}$.]



42. Three radii are drawn in a circle of radius $2\,a$, so as to divide the circumference into three equal parts; and, with the middle of these radii as centers, arcs are drawn, each with the radius a, so as to form a closed figure (trefoil). Show that the length of the perimeter of the trefoil is equal to that of the circle, and find its area.

SOLID GEOMETRY

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[The references are to articles.]

SOLID GEOMETRY

BOOK VI

800. Definitions. A *plane surface* has been defined to be a surface such that if any two of its points be joined by a straight line, that line will lie wholly in the surface.

It follows from this definition that *if a line has two of its points in a plane, it lies wholly in that plane.*

Let it be granted that through any straight line a plane may be passed, and that it may be revolved about the line as an axis.

The plane, in the course of its revolution, takes an infinite number of different positions, from which we infer that *an infinite number of planes can be passed through a straight line.*

The plane of revolution is infinite in extent and in the course of one revolution on its axis takes in every point in the universe.

A plane is said to be *determined* by certain points and lines if it is the only plane that contains those points and lines.

Proposition I. Theorem

801. *A plane is determined*

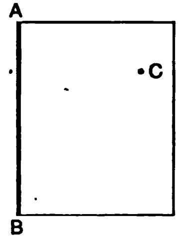
I. *By a straight line and a point without the line.*

II. *By three points not in the same straight line.*

III. *By two intersecting lines.*

IV. *By two parallel lines.*

I. **Let** *AB* be any line, and *C* a point without.



To Prove that *AB* and *C* determine a plane.

Proof. Pass a plane through *AB* and let it be revolved about *AB* as an axis. (§ 800.)

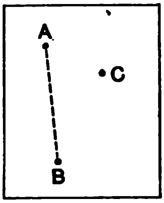
In one position it will take in the point *C*.

Therefore a plane can be passed through a straight line and a point without.

Now if the plane be revolved about *AB* in either direction, it will no longer contain the point *C* until it reaches its original position.

Therefore only one plane can be passed through a straight line and a point without. Q.E.D.

II. **Let** *A*, *B*, and *C* be three points not in the same straight line.



To Prove that they determine a plane.

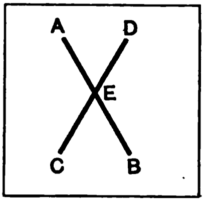
Proof. Join *A* and *B*.

Through the line *AB* and the point *C*, one plane and only one can be passed. (?)

Therefore through the three points *A*, *B*, and *C*, one plane and only one can be passed. Q.E.D.

III. **Let** *AB* and *CD* be two intersecting lines.

To Prove that they determine a plane.



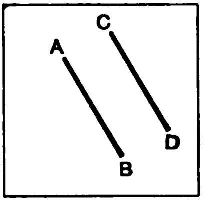
Proof. A plane can be passed through the line *AB* and the point *C*. (?)

DC has two of its points, *C* and *E*, in this plane.

∴ *DC* lies wholly in this plane.

A plane can therefore be passed through *AB* and *CD*.

Since only one plane can be passed through *AB* and the point *C*, only one plane can be passed through *AB* and *CD*. Q.E.D.



IV. **Let** AB and CD be two parallel lines.

To Prove that they determine a plane.

Proof. Since by definition, § 107, parallel lines are lines that lie in the same plane and never meet, AB and CD lie in the same plane.

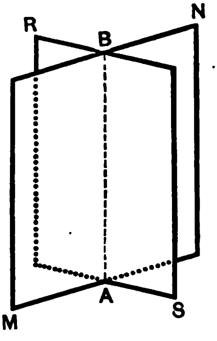
Since only one plane can be passed through AB and the point C, only one plane can be passed through AB and CD. Q.E.D.

802. Exercise. **A straight line can intersect a plane in only one point.**

803. Exercise. **Three intersecting lines, each intersecting the other two, but not in a common point, are in the same plane.**

Proposition II. Theorem

804. *The intersection of two planes is a straight line.*



Let MN and RS be two intersecting planes.

To Prove that their intersection is a straight line.

Proof. Let A and B be any two points common to both planes. Draw AB.

AB lies in the plane MN. (?)

AB also lies in the plane RS. (?)

AB is therefore common to the two planes.

There can be no point without AB common to the two planes. (?)

$\therefore$ AB is the intersection of the two planes. Q.E.D.

805. Exercise. **A plane can cut a circumference in only two points.**

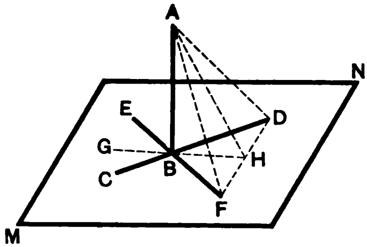
806. Definition. A straight line is perpendicular to a plane if it is perpendicular to every line of the plane that passes through its foot.

PROPOSITION III. THEOREM

807. *If a line is perpendicular to each of two lines at their point of intersection, it is perpendicular to their plane.*

Let *AB* be ⊥ to *CD* and to *EF* at their point of intersection *B*.

To Prove *AB* ⊥ to the plane *MN*.



Proof. Draw *GH*, any other line of the plane *MN* passing through *B*.

Through any point of *GH* as *H* draw *FD*, limited by *CD* and *EF*, and bisected at *H*. (§ 465.)

Draw *AF*, *AH*, and *AD*.

In the $\triangle AFD$, $\overline{AF}^2 + \overline{AD}^2 = 2\,\overline{AH}^2 + 2\,\overline{FH}^2$. (?) (1)

In the $\triangle BFD$, $\overline{BF}^2 + \overline{BD}^2 = 2\,\overline{BH}^2 + 2\,\overline{FH}^2$. (?) (2)

Subtract (2) from (1).

$$\overline{AF}^2 - \overline{BF}^2 + \overline{AD}^2 - \overline{BD}^2 = 2\,\overline{AH}^2 - 2\,\overline{BH}^2.$$

$$\overline{AB}^2 + \overline{AB}^2 = 2\,\overline{AH}^2 - 2\,\overline{BH}^2. \quad (?)$$

$$2\,\overline{AB}^2 = 2\,\overline{AH}^2 - 2\,\overline{BH}^2.$$

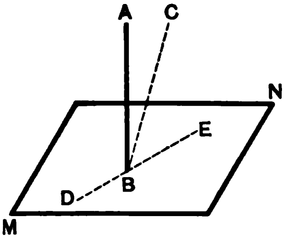
$$\overline{AB}^2 = \overline{AH}^2 - \overline{BH}^2.$$

$$\overline{AB}^2 + \overline{BH}^2 = \overline{AH}^2.$$

∴ $\triangle ABH$ is right-angled (§ 663) and *AB* is ⊥ to *GH*.

Since *GH* is any line of the plane *MN* passing through *B*, *AB* is perpendicular to every line of the plane passing through *B*, and by definition *AB* is perpendicular to the plane. Q.E.D.

808. COROLLARY I. *At a point on a plane only one perpendicular can be erected to the plane.*



Let AB be ⊥ to MN at B.

To Prove that AB is the only ⊥ that can be erected to MN at B.

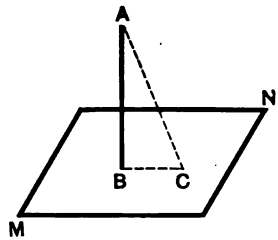
Proof. Suppose a second ⊥ to be erected, as BC.

Pass a plane through AB and BC. (?)

This plane will intersect MN in a straight line DE. (?)

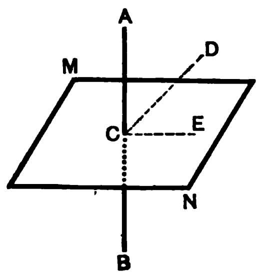
Then AB and BC, lying in the same plane, are both ⊥ to a line of that plane at the same point, which is contradictory to (?) etc.

809. COROLLARY II. *From a point without a plane only one perpendicular can be drawn to the plane.*



Suggestion. Suppose a second ⊥ could be drawn from the given point. Pass a plane through the two ⊥'s and proceed as in § 808.

810. COROLLARY III. *All perpendiculars that can be drawn to a line at a given point lie in a plane that is perpendicular to the line at the given point.*



[The plane MN is ⊥ to AB at the point C. Suppose CD is ⊥ to AB and *not* lying in MN. Let the plane of AC and CD cut MN in CE. The ∠ ACE is a R.A. (?) and equals ∠ ACD, which is absurd.]

811. Exercise. Through a given point of a line pass a plane perpendicular to the line.

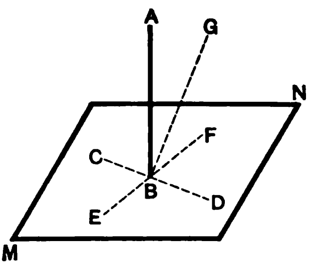
812. Exercise. Through a given point without a line pass a plane perpendicular to the line.

813. Exercise. Find the locus of points in space that are each equally distant from two given points.

Proposition IV. Problem

814. *From a given point to draw a perpendicular to a plane.*

I. Let *MN* be the plane, and *B* be the given point in the plane.



Required to erect a ⊥ to *MN* at *B*.

Draw any line in the plane *MN* through *B*, as *CD*.

Draw *EF* in the plane *MN* ⊥ to *CD*.

Pass any plane, other than *MN*, through *CD*, and in this plane draw *BG* ⊥ to *CD*.

Pass a plane through *EF* and *BG*, and in this plane draw *AB* ⊥ to *EF*.

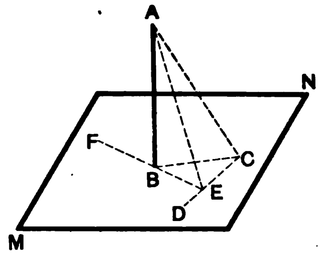
Then is *AB* ⊥ to *MN*.

CD, being ⊥ to *EF* and *BG*, is ⊥ to their plane. (?)

∴ *CD* is ⊥ to *AB*. (?)

AB, being ⊥ to *CD* and *EF*, is ⊥ to *MN*. Q.E.F.

II. **Let** *MN* be the plane, and *A* be the given point without the plane.



Required to draw a ⊥ to *MN* from *A*.

Draw any line as *DC* in the plane *MN*.

Pass a plane through *DC* and *A*, and in this plane draw *AE* ⊥ to *DC*.

Draw *FE* ⊥ to *DC* and in the plane *MN*.

Pass a plane through *FE* and *EA*, and in this plane draw *AB* ⊥ to *FE*.

Then is *AB* ⊥ to *MN*.

From any point on *DC*, as *C*, draw *CB* and *CA*.

△*ABE*, *BEC*, and *AEC* are R. A. △.

$$\overline{AB}^2 + \overline{BE}^2 = \overline{AE}^2. \quad (?) \qquad (1)$$

$$\overline{AE}^2 + \overline{EC}^2 = \overline{AC}^2. \quad (?) \qquad (2)$$

$$\overline{BC}^2 = \overline{BE}^2 + \overline{EC}^2. \quad (?) \qquad (3)$$

Add (1), (2), and (3). $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$.

∴ *AB* is ⊥ to *BC*. (?)

AB is ⊥ to *MN*. (?) Q.E.F.

815. Exercise. If from a point without a plane perpendiculars are drawn to different lines of the plane, and from the feet of the perpendiculars lines are drawn in the plane and perpendicular to the lines of the plane, the last lines drawn will be concurrent.

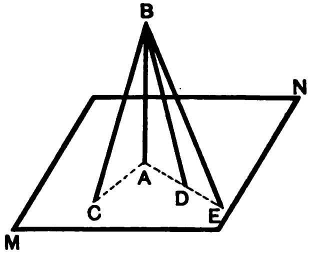
PROPOSITION V. THEOREM

816. *If from a point without a plane a perpendicular is drawn to the plane and oblique lines are drawn to different points of it,*

I. *The perpendicular is shorter than any oblique line.*

II. *Two oblique lines that meet the plane at equal distances from the foot of the perpendicular are equal.*

III. *Of two oblique lines that meet the plane at points unequally distant from the foot of the perpendicular, the one at the greater distance is the longer.*



I. **Let** B be the point without the plane MN, BA be the $\perp$ to MN, and BC any oblique line.

To Prove $BA < BC$.

Proof. See proof of § 216.

II. **Let** BC and BD meet MN at points equally distant from A.

To Prove $BC = BD$.

[Proof is left to the student.]

III. **Let** $AE > AC$.

To Prove $BE > BC$.

Proof. See proof of § 216.

817. Corollary. *If from a point without a plane a perpendicular and two equal oblique lines are drawn to the plane, the points in which the oblique lines meet the plane are equally distant from the foot of the perpendicular.*

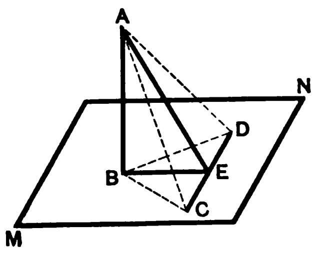
818. Exercise. If from a point without a plane a number of equal oblique lines are drawn to the plane, the points in which they meet the plane are on the circumference of a circle; and the line joining the center of this circle with the given point without the plane is perpendicular to the plane.

819. Exercise. What is the locus of points in space each equally distant from the vertices of a triangle?

820. Exercise. If a line meets a plane and makes equal angles with each of three lines of that plane, it is perpendicular to the plane.

Proposition VI. Theorem

821. *If from the foot of a perpendicular to a plane a line is drawn at right angles to any line of the plane, and the point of intersection is joined with any point of the perpendicular to the plane, the last line drawn is perpendicular to the line of the plane.*



Let AB be $\perp$ to the plane MN, and BE be $\perp$ to any line CD in the plane MN, and EA drawn from E to any point of AB.

To Prove $AE \perp$ to CD.

Proof. Lay off $EC = ED$, and draw BC, BD, AC, and AD.

Prove $BC = BD$.

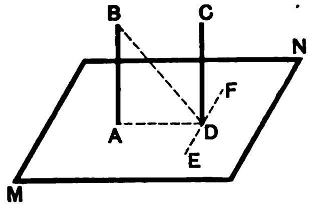
$AC = AD$. (§ 816.)

Then AE is $\perp$ to CD. (?) Q.E.D.

822. Exercise. If AB is perpendicular to the plane MN (see figure of § 821) and AE is perpendicular to DC, a line of the plane MN, prove that BE is perpendicular to DC.

PROPOSITION VII. THEOREM

823. *Two perpendiculars to the same plane are parallel.*



Let AB and CD be $\perp$ to the plane MN.

To Prove $AB \parallel$ to CD.

Proof. Draw AD. Draw EF in the plane MN and $\perp$ to AD. Join D with any point of AB, as B.

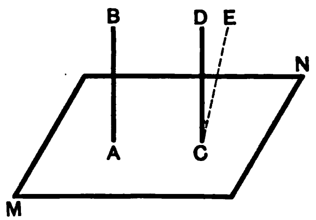
BD is $\perp$ to EF. (?)

Since AD, BD, and CD are $\perp$ to EF, they are in the same plane. (?)

AB also lies in this plane. (?)

AB and CD, lying in the same plane and being both $\perp$ to AD, are parallel. Q.E.D.

824. COROLLARY. *If one of two parallels is perpendicular to a plane, the other is also.*



Let AB be $\parallel$ to CD and AB be $\perp$ to the plane MN.

To Prove $CD \perp$ to MN.

Proof. Suppose CD is not $\perp$ to MN, and draw $CE \perp$ to MN.

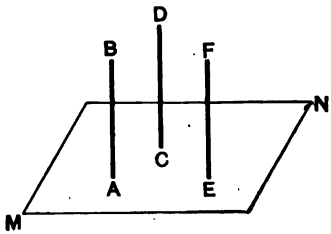
CE is ∥ to *AB*. (?)
This contradicts (?).
∴ *CD* is ⊥ to *MN*. Q.E.D.

825. Exercise. If a plane is passed through a diagonal of a parallelogram, the perpendiculars to this plane from the extremities of the other diagonal are equal.

826. Exercise. If perpendiculars to a plane are drawn from the vertices of a quadrilateral lying without the plane, their sum is equal to four times the perpendicular drawn from the middle point of the line which joins the centers of the diagonals of the quadrilateral.

Proposition VIII. Theorem

827. *If two lines are parallel to a third line, they are parallel to each other.*



Let *AB* and *CD* be ∥ to *EF*.

To Prove *AB* ∥ to *CD*.

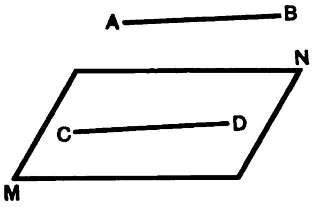
Proof. Pass the plane *MN* ⊥ to *EF*.
AB and *CD* are ⊥ to *MN*. (?)
AB is ∥ to *CD*. (?) Q.E.D.

828. Exercise. In a *gauche* (pro. gōsh) quadrilateral (a quadrilateral whose sides are not in the same plane) the lines joining the middle points of the sides form a parallelogram.

829. Definition. A straight line and a plane are parallel to each other, if they never meet how far soever they are produced.

Proposition IX. Theorem

830. *If a straight line is parallel to a line of a plane, it is parallel to the plane.*



Let *AB* be ∥ to *CD* of the plane *MN*.

To Prove *AB* ∥ to *MN*.

Proof. *AB* and *CD* are in the same plane. **(?)**

AB cannot meet *MN* unless it meets *CD*.

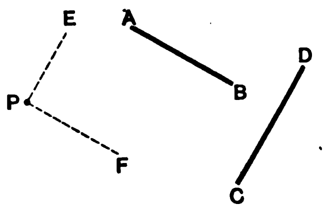
But *AB* and *CD* are parallel.

∴ *AB* cannot meet *MN*, and *AB* and *MN* are parallel. Q.E.D.

831. Corollary I. *Through a given line pass a plane parallel to a given line.*

832. Corollary II. *Through a given point pass a plane parallel to two lines in space.*

[Let *P* be the given point, and *AB* and *CD* be two lines in space. Draw *PE* ∥ to *CD*, and *PF* ∥ to *AB*. Prove that the plane passed through *PE* and *PF* is ∥ to *AB* and *CD*.]



833. Corollary III. *If a line is parallel to a plane, the intersection of this plane with any plane through the line is parallel to the line.*

834. Exercise. Through a point without a plane draw a line parallel to the plane. How many such parallels can be drawn?

835. Exercise. Through a point in a plane draw a line in the plane and parallel to a line in space. When is this possible?

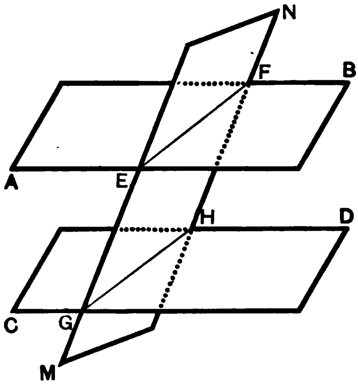
836. Exercise. Through a point in a plane draw a line in the plane and parallel to another plane. When can an infinite number of such parallels be drawn? When but one?

837. Exercise. If two planes that intersect contain two lines that are parallel, the intersection of the planes is parallel to the lines.

838. Definition. Two planes are parallel if they never meet how far soever they are produced.

Proposition X. Theorem

839. *If a plane intersects two parallel planes, the lines of intersection are parallel.*



Let the plane *MN* intersect the parallel planes *AB* and *CD* in the lines *EF* and *GH*.

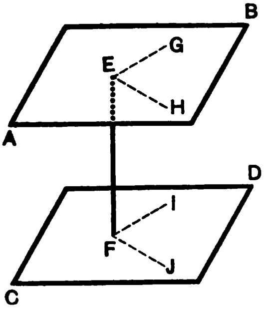
To Prove *EF* and *GH* parallel.

Proof. *EF* and *GH* lie in the plane *MN*, and they cannot meet, otherwise the planes *AB* and *CD* would meet.

∴ *EF* and *GH* are parallel. Q.E.D.

PROPOSITION XI. THEOREM

840. *If a line is perpendicular to one of two parallel planes, it is perpendicular to the other also.*



Let *AB* and *CD* be two parallel planes and *EF* be ⊥ to *AB*.

To Prove *EF* ⊥ to *CD*.

Proof. Pass any two planes through *EF*, cutting *AB* in *EG* and *EH*, and *CD* in *FI* and *FJ*.

EG is ∥ to *FI* (?), and *EH* is ∥ to *FJ*. (?)

EF, being ⊥ to *EG*, is also ⊥ to *FI*.

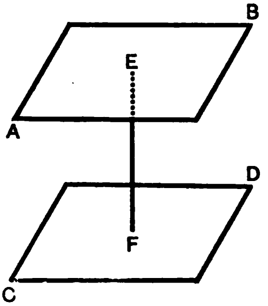
EF, being ⊥ to *EH*, is also ⊥ to *FJ*.

∴ *EF* is ⊥ to the plane *CD*. Q.E.D.

841. EXERCISE. **If two intersecting straight lines are each parallel to a given plane, their plane is parallel to the given plane.**

PROPOSITION XII. THEOREM

842. *Two planes that are perpendicular to the same line are parallel.*



Let the planes *AB* and *CD* be ⊥ to *EF*.

To Prove *AB* ∥ to *CD*.

Proof. If *AB* and *CD* are not parallel, they will meet.

Join any point of their line of intersection with *E* and *F*.

Show that both of these lines are ⊥ to *EF*, which contradicts (?).

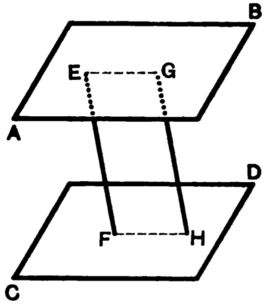
∴ *AB* is ∥ to *CD*. Q.E.D.

843. Corollary. *Through a given point pass a plane parallel to a given plane.*

844. Exercise. If two planes are parallel to a third plane, they are parallel to each other. [Use §§ 840, 842.]

Proposition XIII. Theorem

845. *Parallel lines included between parallel planes are equal.*



Let EF and GH be two parallel lines included between the parallel planes AB and CD.

To Prove $EF = GH$.

Proof. Pass a plane through EF and GH.

Its intersections with AB and CD are EG and FH respectively.

Prove $EGHF$ a parallelogram.

Whence $EF = GH$. Q.E.D.

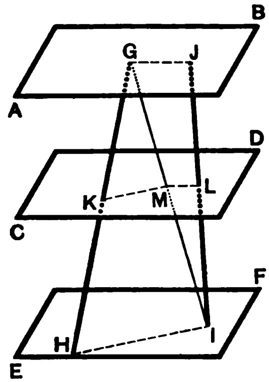
846. Corollary. *Two parallel planes are everywhere equally distant.*

847. Exercise. What is the locus of points in space at a given distance from a given plane?

848. Exercise. What is the locus of points in space equally distant from two parallel planes?

Proposition XIV. Theorem

849. *If two straight lines intersect three parallel planes, their corresponding segments are proportional.*



Let GH and JI intersect the three parallel planes AB, CD, and EF.

To Prove $$\frac{GK}{JL}=\frac{KH}{LI}.$$

Proof. Draw GI. Pass a plane through GH and GI, and let KM and HI be the intersections of this plane with CD and EF.

KM is parallel to HI. (?)

Whence $$\frac{GK}{KH}=\frac{GM}{MI}.$$

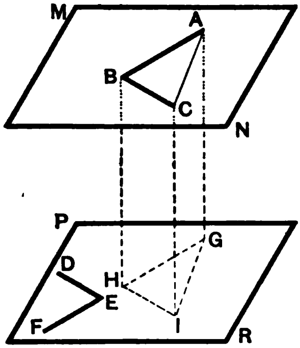
Similarly, prove $$\frac{JL}{LI}=\frac{GM}{MI}.$$

Whence $$\frac{GK}{KH}=\frac{JL}{LI}, \text{ or } \frac{GK}{JL}=\frac{KH}{LI}.$$ Q.E.D.

850. Corollary. *If two straight lines intersect any number of parallel planes, their corresponding segments are proportional.*

PROPOSITION XV. THEOREM

851. *If two angles have their sides parallel, right side to right side and left side to left side, the angles are equal and their planes are parallel.*



Let the angles ABC and DEF, lying in the planes MN and PR, respectively, have $BA \parallel$ to EF and $BC \parallel$ to ED.

To Prove $\angle ABC = \angle DEF$ and their planes parallel.

Proof. From A, B, and C draw AG, BH, and CI respectively $\perp$ to PR.

Show that AG, BH, and CI are equal and parallel.

$\triangle$s ABC and GHI are equal in all respects. (?)

Whence $\angle ABC = \angle GHI$.

DE is $\parallel$ to HI (?) and FE is $\parallel$ to GH. (?)

$\therefore \angle DEF = \angle GHI$. (?)

Therefore $\angle ABC = \angle DEF$.

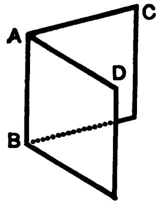
Show that the planes MN and PR are both $\perp$ to BH and are parallel. Q.E.D.

852. COROLLARY. *If two angles have their sides parallel, right side to left side and left side to right side, the angles are supplementary and their planes are parallel.*

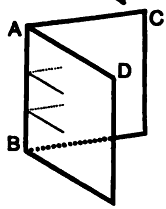
853. Exercise. Through two lines not in the same plane pass parallel planes.

854. Definitions. A *dihedral angle* is the amount of divergence of two planes that meet.

The line of meeting (§ 804) is called the *edge* of the dihedral angle, and the two planes are its *faces*.

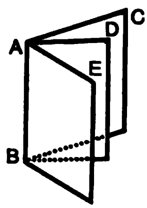


A dihedral angle may be designated by the two letters on its edge, or if more than one dihedral angle have the same edge, by the two letters on the edge together with an additional letter on each face. The above may be read angle *AB*, or angle *C–AB–D*.



The *plane angle* of a dihedral angle is an angle formed by lines in the two faces perpendicular to the edge at the same point.

The plane angles of a dihedral angle are all equal (§ 851).

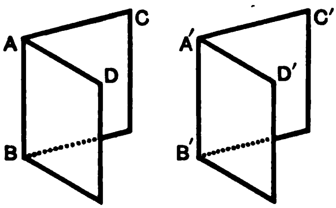


Two dihedral angles are *adjacent* if they have a common edge and the same face between them. [The dihedral angles *C–AB–D* and *D–AB–E* are adjacent.]

Two dihedral angles are *vertical* if the faces of one are the prolongations of those of the other.

Proposition XVI. Theorem.

855. *Two dihedral angles are equal if their plane angles are equal.*



Let the dihedral angles AB and $A'B'$ have their plane angles CAD and $C'A'D'$ equal.

To Prove that the dihedral angles AB and $A'B'$ are equal.

Proof. Place the dihedral angle AB so that its plane angle CAD shall coincide with the plane angle $C'A'D'$. AB, being $\perp$ to the plane of $\angle CAD$, will coincide with $A'B'$. (?)

The faces of the dihedral angle AB will coincide with the faces of $A'B'$. (?)

$\therefore$ the dihedral angles AB and $A'B'$ are equal. Q.E.D.

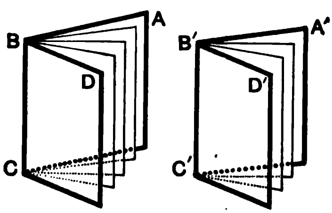
CONVERSELY. If two dihedral angles are equal, their plane angles are equal. [The dihedral angles may be made to coincide, and the plane angles also.]

856. EXERCISE. **If two planes intersect each other, the opposite or vertical dihedral angles are equal.**

PROPOSITION XVII. THEOREM

857. *Dihedral angles are proportional to their plane angles.*

CASE I. When the plane angles are commensurable.



Let $A\text{–}BC\text{–}D$ and $A'\text{–}B'C'\text{–}D'$ be two dihedral angles, having their plane angles ABD and $A'B'D'$ commensurable.

To Prove $$\frac{A\text{–}BC\text{–}D}{A'\text{–}B'C'\text{–}D'} = \frac{ABD}{A'B'D'}.$$

Proof. Suppose that the common unit of measure of angles ABD and $A'B'D'$ is contained in $\angle ABD$ m times, and in $\angle A'B'D'$ n times.

Then $$\frac{ABD}{A'B'D'} = \frac{m}{n}.$$

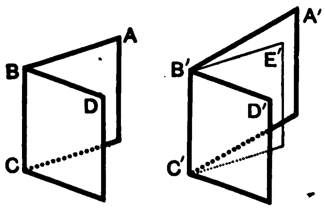
By passing planes through the edge BC and each side of the plane angles, the dihedral angle $A\text{–}BC\text{–}D$ may be divided into m dihedral angles. Similarly $A'\text{–}B'C'\text{–}D'$ may be divided into n dihedral angles.

The dihedral angles thus formed are all equal to each other (§ 855).

Then $$\frac{A\text{–}BC\text{–}D}{A'\text{–}B'C'\text{–}D'} = \frac{m}{n}.$$

Whence $$\frac{A\text{–}BC\text{–}D}{A'\text{–}B'C'\text{–}D'} = \frac{ABD}{A'B'D'}.$$ Q.E.D.

CASE II. When the plane angles are incommensurable.



Let $A\text{–}BC\text{–}D$ and $A'\text{–}B'C'\text{–}D'$ be two dihedral angles whose plane angles ABD and $A'B'D'$ are incommensurable.

To Prove $$\frac{A\text{–}BC\text{–}D}{A'\text{–}B'C'\text{–}D'} = \frac{ABD}{A'B'D'}.$$

Proof. Let ABD be divided into a number of equal angles, and one of these be applied to $\angle A'B'D'$ as a unit of measure.

$\angle A'B'D'$ will not contain this unit of measure exactly, but a certain number of these angles will extend as far as $E'B'D'$, leaving a remainder $\angle E'B'A'$ smaller than the unit of measure.

$$\frac{E'-B'C'-D'}{A-BC-D} = \frac{\angle E'B'D'}{\angle ABD}. \quad (?)$$

By increasing indefinitely the number of equal parts into which $\angle ABD$ is divided, the divisions will become smaller and smaller, and the remainder $\angle E'B'A'$ will also diminish indefinitely.

$\frac{E'-B'C'-D'}{A-BC-D}$ and $\frac{\angle E'B'D'}{\angle ABD}$ are variables, and they are always equal to each other. (?)

The limit of $\frac{E'-B'C'-D'}{A-BC-D}$ is $\frac{A'-B'C'-D'}{A-BC-D}$, and the limit of $\frac{\angle E'B'D'}{\angle ABD}$ is $\frac{A'B'D'}{ABD}$.

$$\therefore \frac{A'-B'C'-D'}{A-BC-D} = \frac{A'B'D'}{ABD}. \quad (?)$$

Q.E.D.

858. COROLLARY. *The plane angles of dihedral angles may be taken as their measures.*

859. EXERCISE. If a plane meets another plane, the sum of the adjacent dihedral angles formed is equal to two right angles.

860. EXERCISE. If a plane intersects two parallel planes, the sum of the interior dihedral angles on the same side is equal to two right angles; the corresponding angles are equal, etc.

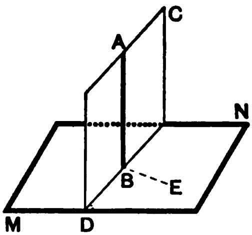
861. EXERCISE. Show that the converse of the preceding exercise is not necessarily true.

862. EXERCISE. The planes that bisect two supplementary adjacent dihedral angles are perpendicular to each other.

863. EXERCISE. The planes that bisect two vertical dihedral angles form one and the same plane.

PROPOSITION XVIII. THEOREM

864. *If a straight line is perpendicular to a plane, every plane passed through the line is also perpendicular to the plane.*



Let AB be ⊥ to MN, and CD be any plane through AB.

To Prove CD ⊥ to MN.

Proof. Draw BE in the plane MN and ⊥ to DB.
AB is ⊥ to DB and also to BE. (?)
∠ ABE is a R. A., and it is the measure of the angle $A–BD–E$.
∴ CD is ⊥ to MN. Q.E.D.

865. COROLLARY. *Through any given line pass a plane perpendicular to a given plane. How many such planes can be passed?*

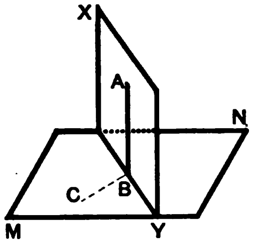
866. EXERCISE. If a line is parallel to one plane and perpendicular to another, the two planes are perpendicular to each other.

867. EXERCISE. A plane that is perpendicular to the edge of a dihedral angle is perpendicular to both of its faces.

868. EXERCISE. If three lines are perpendicular to each other at a common point, each line is perpendicular to the plane of the other two, and the planes of the lines are perpendicular to each other.

Proposition XIX. Theorem

869. *If two planes are perpendicular to each other and a line is drawn in one of them perpendicular to their line of intersection, it is perpendicular to the other plane.*



Let *XY* be ⊥ to *MN*, and *AB* (in the plane *XY*) be ⊥ to *BY*.

To Prove *AB* ⊥ to *MN*.

Proof. Draw *BC* in *MN* and ⊥ to *BY*.

Show that ∠ *ABC* is the measure of the dihedral angle *A–BY–C*.

Since the planes are ⊥ to each other, ∠ *ABC* is a R. A.

AB, being ⊥ to both *BC* and *BY*, is ⊥ to *MN*. Q.E.D.

870. Corollary. *If two planes are perpendicular to each other and a line is drawn perpendicular to one of the planes at a point in their line of intersection, it lies in the other plane.*

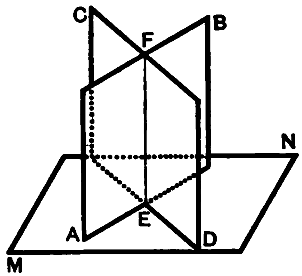
[If not, draw a line in the other plane, ⊥ to line of intersection at the point. Show that there are two perpendiculars to the plane at the same point.]

871. Exercise. If two planes are perpendicular to each other, a perpendicular to one of them from any point of the other will lie in the other plane.

[In figure of § 869 suppose that a ⊥ from *A* to *MN* does not lie in *XY*. Draw *AB* ⊥ to *BY*.]

Proposition XX. Theorem

872. ***If two intersecting planes are each perpendicular to a third plane, their line of intersection is perpendicular to that plane.***



Let the intersecting planes *AB* and *CD* be ⊥ to *MN*.

To Prove that their line of intersection is ⊥ to *MN*.

Proof. At *E*, the point common to the three planes, erect a ⊥ to *MN*.

This ⊥ lies in both *AB* and *CD*. (?)

It is therefore their line of intersection.

Their line of intersection is perpendicular to *MN*. Q.E.D.

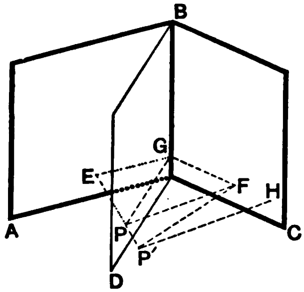
873. Exercise. From any point within a dihedral angle draw a perpendicular to each of its two faces, and show that the angle contained by these perpendiculars is the supplement of the dihedral angle.

874. Exercise. If a plane is perpendicular to each of two intersecting planes, it is perpendicular to their intersection.

875. Exercise. If the intersections of several planes are parallel, the perpendiculars drawn from a common point to the planes lie in the same plane.

PROPOSITION XXI. THEOREM

876. *Any point on the plane that bisects a dihedral angle is equally distant from the faces of the angle, and any point without the bisecting plane is unequally distant from the faces of the angle.*



I. **Let** the plane BD bisect the dihedral angle formed by the planes AB and BC, and let P be any point of BD.

To Prove P equally distant from the faces AB and BC.

Proof. Draw PE and PF perpendicular respectively to the planes AB and BC.

Pass a plane through PE and PF, and let EG, PG, and GF be the intersections of this plane with AB, BD, and BC respectively.

The plane EF is perpendicular to AB and BC. (?)

BG is perpendicular to the plane EF. (?)

△ PEG and PFG are equal in all respects. (?)

Whence $$PE = PF.$$ Q.E.D.

II. **Let** P' be any point without the bisecting plane BD.

To Prove P' unequally distant from AB and BC.

Proof. Draw $P'E$ and $P'H \perp$ to AB and BC respectively.

Pass a plane through $P'E$ and $P'H$ intersecting AB, BD, and BC in EG, GP, and GF respectively. [Proof similar to that of § 230.]

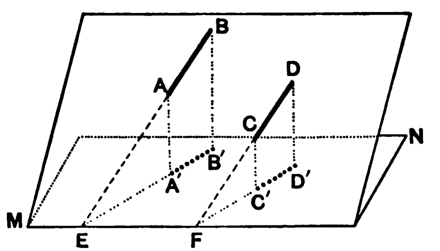
877. Exercise. **What is the locus of points in space that are equally distant from two intersecting planes?**

878. Definitions. The *projection of a point on a plane* is the foot of the perpendicular from the point to the plane.

The *projection of a line on a plane* is the locus of the projections of its points on the plane.

879. Exercise. **The projection of a straight line on a plane is the straight line joining the projections of its extremities on the plane.**

880. Exercise. **Parallel straight lines project into parallel straight lines of proportional length.**



Prove $\angle B = \angle D$ and the planes EBB' and FDD' parallel.
Whence $A'B'$ and $C'D'$ are parallel. (?)

$$\frac{A'B'}{AB} = \frac{EB'}{EB}. \quad (?)$$

$$\frac{C'D'}{CD} = \frac{FD'}{FD}. \quad (?)$$

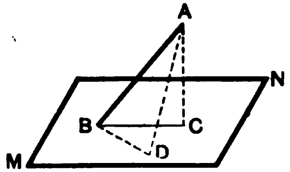
$$\frac{EB'}{EB} = \frac{FD'}{FD}. \quad (?)$$

Whence $$\frac{A'B'}{AB} = \frac{C'D'}{CD}, \text{ or } \frac{A'B'}{C'D'} = \frac{AB}{CD}.$$ Q.E.D.

881. Definition. The angle made by a line with a plane is understood to be the acute angle that it makes with its projection on the plane.

Proposition XXII. Theorem

882. *The angle made by a line with a plane is the least angle made by that line with any line of the plane.*



Let ABC be the angle made by AB with the plane MN, and BD be any line of MN (other than the projection of AB) passing through B.

To Prove $\angle ABC < \angle ABD$.

Proof. Lay off $BD = BC$, and draw AD.

$AC < AD$. (?)

$\therefore \angle ABC < \angle ABD$. (?) Q.E.D.

883. Exercise. If two parallel lines meet a plane, they make equal angles with it.

State the converse of this exercise. Show that it is not necessarily true.

884. Exercise. The angles made by a line with two parallel planes are equal.

Is the converse of this exercise necessarily true? For what angles will this converse and the converse of § 883 always be true?

Proposition XXIII. Problem

885. *To draw a line perpendicular to any two lines.*

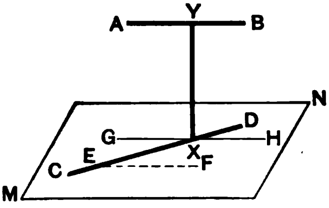
Case I. When the two given lines are parallel.

[Proof is left for the student.]

Case II. When the two lines intersect.

[Proof is left for the student.]

CASE III. When the two lines do not lie in the same plane.



Let AB and CD be any two lines not situated in the same plane.

Required to draw a line ⊥ to both AB and CD.

Draw EF ∥ to AB through any point of CD.

Pass a plane MN through CD and EF.

AB is ∥ to MN. (?)

Pass a plane through AB and ⊥ to MN. (?)

Let GH be the line of intersection of this plane with MN.

AB is ∥ to GH. (?)

At the point of intersection of CD and GH draw XY in the plane of AB and GH, and ⊥ to GH.

XY is ⊥ to the plane MN. (?)

XY is ⊥ to CD. (?)

XY is ⊥ to AB. (?) Q.E.F.

886. COROLLARY I. *XY is the only perpendicular to AB and CD.*

887. COROLLARY II. *XY is the shortest distance between AB and CD.*

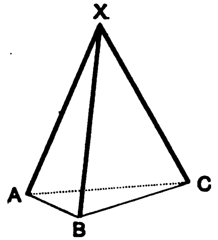
[Join AB and CD by any other line. Pass a plane through AB ∥ to MN. XY is ⊥ to both planes and may be shown to be less than any other line joining the planes and not perpendicular to them.]

888. EXERCISE. If a plane be passed through the middle point of XY parallel to AB and CD (see figure of § 885), it will bisect all lines joining AB and CD.

889. Exercise. What is the locus of points that are equally distant from two lines not in the same plane?

Suggestion. At the middle point of XY (see figure of § 885) draw lines parallel to AB and to CD. Show that the bisectors of the angles formed by these parallels are the required loci.

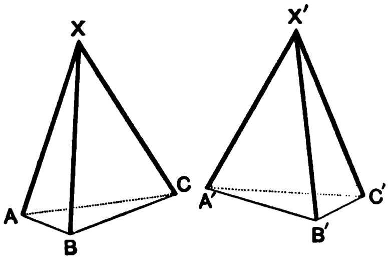
890. Definitions. A *polyhedral* angle is a figure formed by three or more planes meeting in a common point.



The point of meeting is called the *vertex* of the angle. The lines in which the planes meet are its *edges*, and the portions of the planes lying between these edges are its *faces*. The plane angles in the faces at the vertex are called the *face angles* of the polyhedral angle.

A *trihedral angle* is a polyhedral angle having three faces.

If the edges of a polyhedral angle are prolonged through the vertex, the new polyhedral angle formed is called the *vertical angle* of the first angle.



Two polyhedral angles are equal if they can be placed so that their edges coincide.

Two polyhedral angles are symmetrical if the face angles and the dihedral angles of the one are equal respectively to the face angles and the dihedral angles of the other, but taken in reverse order.

[In general, symmetrical angles cannot be made to coincide.]

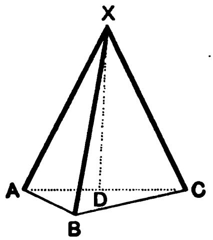
A polyhedral angle is *convex* if the section made by a plane cutting all of its edges is a convex polygon.

891. Exercise. Show that two vertical polyhedral angles are symmetrical.

892. Exercise. Show that a plane may be passed perpendicular to only one edge, and to only two faces of a polyhedral angle.

Proposition XXIV. Theorem

893. *The sum of two face angles of a trihedral angle is greater than the third.*



Let $X\text{–}ABC$ be any trihedral angle.

To Prove $\angle AXB + \angle BXC > \angle AXC.$

Proof. If $\angle AXC$ is equal to or less than either $\angle AXB$ or $\angle BXC$, the proposition is self-evident. Suppose $\angle AXC$ is greater than either of the other two angles.

Draw XD in the plane AXC and making $\angle AXD = \angle AXB.$

Draw AC. Lay off $XB = XD$, and draw AB and CB.

$$\triangle AXB = \triangle AXD \text{ (?), whence } AD = AB.$$

$$AB + BC > AD + DC \text{ (?) and } BC > DC.$$

$$\therefore \angle BXC > \angle DXC. \quad \text{(?)} \quad (a)$$

By construction $\angle AXB = \angle AXD. \quad (b)$

Adding (a) and (b) member to member,

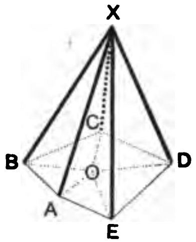
$$\angle AXB + \angle BXC > AXC. \qquad \text{Q.E.D.}$$

894. Exercise. Show that any face angle of a polyhedral angle is less than the sum of the other face angles.

895. Exercise. The sum of the angles of a gauche quadrilateral is less than four right angles.

PROPOSITION XXV. THEOREM

896. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Let $X\text{-}ABCDE$ be a polyhedral angle whose edges are cut by any plane in the points A, B, C, D, E.

To Prove the sum of the face angles of the polyhedral angle less than 4 R.A.'s.

Proof. Connect any point O in the polygon $ABCDE$ with the vertices.

The number of △ with the common vertex O is the same as the number having X for a vertex. (?)

Since $\angle XBA + \angle XBC > \angle ABO + \angle OBC,$ (?)

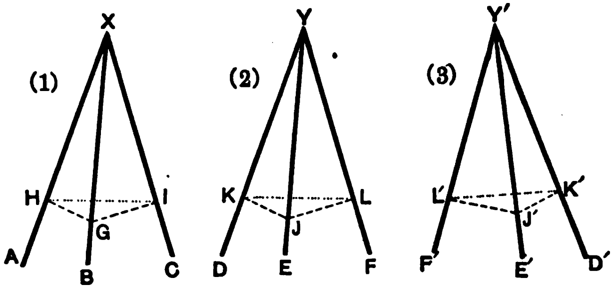
and $\angle XAB + \angle XAE > \angle BAO + \angle OAE,$ (?) etc.,

the sum of the base angles of the triangles having X for a common vertex is greater than the sum of the base angles of the triangles having O for a vertex. Therefore the sum of the angles about the vertex X is less than the sum of the angles about O, *i.e.* less than four right angles. Q.E.D.

897. EXERCISE. What is the greatest number of equilateral triangles that can be grouped about a point so as to form a convex polyhedral angle?

Proposition XXVI. Theorem

898. *If two trihedral angles have their face angles equal each to each, their corresponding dihedral angles are equal.*



Let the trihedral angles $X\text{–}ABC$ and $Y\text{–}DEF$ have $\angle AXB = \angle DYE$, $\angle BXC = \angle EYF$, and $\angle AXC = \angle DYF$.

To Prove that their corresponding dihedral angles are equal.

Proof. From any point G on XB draw GI and GH in the planes BXC and BXA respectively, and $\perp$ to XB. Draw HI. Lay off $YJ = XG$ and draw JL and JK in the planes JYL and JYK respectively, and $\perp$ to YE. Draw KL.

△ GXH and JYK are equal (?), whence $GH = JK$ and $XH = YK$.

△ GXI and JYL are equal (?), whence $GI = JL$ and $XI = YL$.

△ HXI and KYL are equal (?), whence $HI = KL$.

△ GHI and JKL are equal (?), whence $\angle HGI = \angle KJL$. Since $\angle HGI$ and $\angle KJL$ are the measures of the dihedral angles whose edges are XB and YE, $\therefore$ the dihedral angles $A\text{–}XB\text{–}C$ and $D\text{–}YE\text{–}F$ are equal.

Similarly the remaining dihedral angles may be proved equal.

Q.E.D.

899. Corollary I. If the equal angles are arranged in the same order in the two figures, as in (1) and (2), the trihedral angles are equal and may be made to coincide. If the equal angles are arranged in reverse order, as in (1) and (3), the trihedral angles are symmetrical. Therefore, *Two trihedral angles that have their face angles equal each to each are either equal or symmetrical.*

900. Corollary II. *Two trihedral angles are equal or symmetrical if they have two face angles and the included dihedral angle of one equal respectively to two face angles and the included dihedral angle of the other.*

901. Exercise. Two trihedral angles are equal or symmetrical if they have two dihedral angles and the included face angle of one equal respectively to two dihedral angles and the included face angle of the other.

902. If two face angles of a trihedral angle are equal, the two opposite dihedral angles are equal.

Suggestion. Pass a plane through the edge common to the two equal angles and the bisector of the remaining face angle. Apply the proposition to the two trihedral angles formed.

EXERCISES

1. Through a given point pass a plane perpendicular to a given plane. [Show that an infinite number of such planes can be passed.]

2. Through a given point pass a plane parallel to a given plane. [How many possible?]

3. A straight line and a plane perpendicular to the same line are parallel.

4. What is the locus of points in space that are each equally distant from all points in the circumference of a circle?

5. From two given points on the same side of a plane draw two lines meeting in the given plane and making equal angles with it.

6. Show that there are an infinite number of pairs of lines answering the conditions of the preceding exercise, and that the points in which they meet the plane are on the circumference of a circle.

7. Find a point on a plane such that the sum of its distances from two given points on the same side of the plane shall be a minimum. [Exercise 5.]

8. If a line is parallel to the intersection of two planes, it is parallel to each of the planes.

9. If a line is parallel to each of two intersecting planes, it is parallel to their intersection.

10. If the sum of two adjacent dihedral angles is two right angles, their exterior faces are in the same plane.

11. If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.

12. If a line is inclined to a plane at an angle of 60°, its projection on the plane is equal to half the line.

13. What is the locus of points in space that are each equally distant from two intersecting lines?

14. Two dihedral angles whose faces are parallel are either equal or supplementary.

15. What is the locus of points in space that are equally distant from two parallel planes?

16. Find a point that is equally distant from four points that are not in the same plane.

17. If perpendiculars to a plane are drawn from the vertices of a parallelogram lying without the plane, their sum is equal to four times the perpendicular to the plane drawn from the point of intersection of the diagonals of the parallelogram.

18. The planes that bisect the three dihedral angles formed by the faces of a trihedral angle meet in a line.

19. The planes that are perpendicular to the faces of a trihedral angle and pass through the bisectors of the face angles meet in a line.

20. The planes that pass through the edges of a trihedral angle and are perpendicular to the opposite faces meet in a line.

21. The planes that pass through the edges of a trihedral angle and through the bisectors of the opposite face angles, meet in a line.

22. In the trihedral angle $X\text{-}ABC$, XD bisects the face angle BXC. Prove that $\angle AXD < \frac{1}{2}(\angle AXB + \angle AXC)$.

BOOK VII

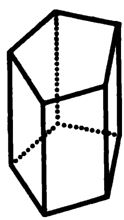
903. Definitions. A *polyhedron* is a solid bounded by planes. The lines of intersection of the bounding planes are the *edges* of the polyhedron, the points of intersection of the edges are its *vertices*, and the portions of the planes included between the edges are its *faces*.

A *diagonal* of a polyhedron is a line joining any two vertices not in the same face.

If the section made by a plane cutting a polyhedron is a convex polygon, the solid is called a *convex polyhedron*.

Only convex polyhedrons are considered in this work.

A polyhedron of four faces is a *tetrahedron*; of six faces, a *hexahedron*; of eight faces, an *octahedron*; of twelve faces, a *dodecahedron*; of twenty faces, an *icosahedron*.



A *prism* is a polyhedron, two of whose faces (called its *bases*) are polygons equal in all respects and having their equal sides parallel, and whose other faces (called its *lateral faces*) are formed by planes passing through the equal sides of the bases.

The lines of intersection of the lateral faces are called the *lateral edges* of the polyhedron.

A prism is *triangular*, *quadrangular*, etc., according as its bases are triangles, quadrilaterals, etc.

The lateral edges are equal and parallel (?), and the lateral faces are parallelograms. (?)

The *altitude* of a prism is the perpendicular distance between its bases.

A *right prism* is a prism whose lateral edges are perpendicular to its bases. Any lateral edge of a right prism is equal to the altitude of the prism.

An *oblique prism* is a prism whose lateral edges are oblique to its bases.

A *regular prism* is a right prism whose bases are regular polygons.

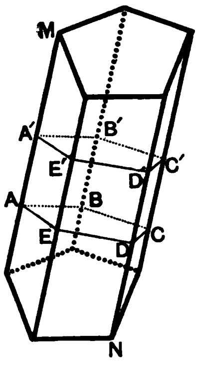
A *right section* of a prism is a section made by a plane perpendicular to the lateral edges of the prism.

The *volume* of a solid is its measure expressed in terms of some other solid taken as the unit of measure.

Two solids are *equivalent* if they have the same volume.

Proposition I. Theorem

904. *The sections of a prism made by parallel planes cutting all the lateral edges are equal polygons.*



Let $ABCDE$ and $A'B'C'D'E'$ be sections of the prism MN made by parallel planes.

To Prove $ABCDE = A'B'C'D'E'$.

Proof. The sides of the polygon $ABCDE$ are parallel to the corresponding sides of $A'B'C'D'E'$. (?)

The polygons $ABCDE$ and $A'B'C'D'E'$ are mutually equilateral. (?)

They are also mutually equiangular. (?)

The polygons $ABCDE$ and $A'B'C'D'E'$ are equal. Q.E.D.

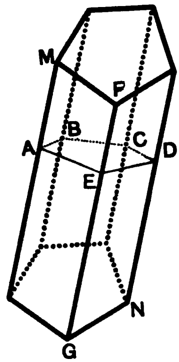
905. Corollary. *A section of a prism parallel to the base is equal to the base.*

906. Exercise. Right sections of a prism are equal.

907. Exercise. A section of a prism made by a plane parallel to a lateral edge is a parallelogram.

Proposition II. Theorem

908. *The lateral area of a prism is equal to the perimeter of a right section multiplied by a lateral edge.*



Let $ABCDE$ be a right section of the prism MN, and FG be any lateral edge.

To Prove the lateral area of $MN =$ perimeter $ABCDE \times FG$.

Proof. The lateral area consists of a number of parallelograms, each of which has a line equal to FG for its base, and one of the sides of $ABCDE$ for its altitude.

The sum of the areas of these parallelograms = the perimeter of $ABCDE \times FG$. Q.E.D.

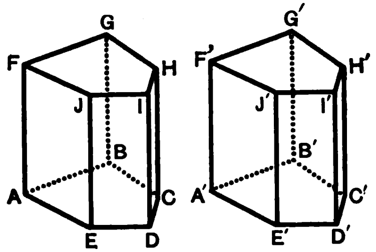
909. Corollary. *The lateral area of a right prism is equal to the perimeter of its base multiplied by a lateral edge.*

910. Exercise. Find the lateral area of a right prism whose altitude is 12 ft. and whose base is a pentagon each side of which is 10 ft.

911. Definition. A truncated prism is that part of a prism included between the base and a section made by a plane not parallel to the base and cutting all the lateral edges.

PROPOSITION III. THEOREM

912. *Two prisms are equal if three faces including a trihedral angle in one are equal respectively to three faces including a trihedral angle in the other, and are similarly placed.*



Let the faces AC, AJ, and AG of the prism AH be equal respectively to the faces $A'C'$, $A'J'$, and $A'G'$ of the prism $A'H'$, and be similarly placed.

To Prove prism $AH =$ prism $A'H'$.

Proof. The trihedral angle $A =$ the trihedral angle A'. (?)

Place the prism $A'H'$ so that trihedral angle A' coincides with trihedral angle A of prism AH.

The face $A'C'$ will coincide with its equal AC; $A'J'$ with AJ, and $A'G'$ with AG.

Since the lateral edges of a prism are parallel and equal, $I'D'$ and $H'C'$ coincide respectively with ID and HC.

Therefore the upper bases coincide, and the prisms coincide throughout and are equal. Q.E.D.

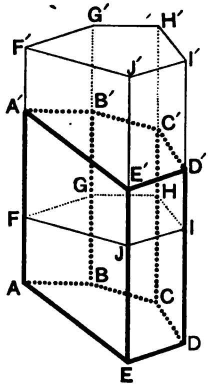
913. COROLLARY I. *Two right prisms having equal bases and equal altitudes are equal.*

914. COROLLARY II. *Two truncated prisms are equal if three faces including a trihedral angle in one are equal respectively to three faces including a trihedral angle in the other, and are similarly placed.*

915. Exercise. **Two triangular prisms are equal, if their lateral faces are equal, each to each, and similarly placed.**

Proposition IV. Theorem

916. *An oblique prism is equivalent to a right prism having for its base a right section of the oblique prism and for its altitude a lateral edge of the oblique prism.*



Let the right prism FH' have for its base a right section of the oblique prism AC' and its altitude FF' equal to each of the lateral edges AA', BB', etc., of the oblique prism.

To Prove the right prism FH' equivalent to the oblique prism AC'.

Proof. The truncated prisms AH and $A'H'$ have their faces $ABCDE$ and $A'B'C'D'E'$ equal. (?)

$AFJE$ and $A'F'J'E'$ have their corresponding sides equal (?), and their corresponding angles are also equal (?)

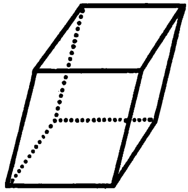
The faces $AFJE$ and $A'F'J'E'$ are therefore equal.

Similarly the faces $AFGB$ and $A'F'G'B'$ are equal.

The truncated prisms AH and $A'H'$ are equal. (§ 914.)

Since $AH + FC' = AC'$ and $A'H' + FC' = FH'$, therefore the oblique prism AC' is equivalent to the right prism FH'. Q.E.D.

917. Definitions. A *parallelopiped* is a prism whose bases are parallelograms.



A *right parallelopiped* is a parallelopiped whose lateral edges are perpendicular to its bases.

A *rectangular parallelopiped* is a right parallelopiped whose bases are rectangles.

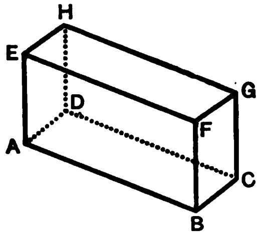
Show that the lateral faces of a right parallelopiped are rectangles, and that the six faces of a rectangular parallelopiped are rectangles.

A *cube* is a parallelopiped whose six faces are squares.

An *oblique parallelopiped* has its lateral edges oblique to its bases.

Proposition V. Theorem

918. *The opposite lateral faces of a parallelopiped are equal and parallel.*



Let AG be any parallelopiped having AC and EG for bases.

To Prove AH and BG equal and parallel.

Proof. AE and BF are equal and parallel. (?)

EH and FG are equal and parallel. (?)

$$\angle AEH = \angle BFG. \quad (?)$$

$$AH = BG. \quad (?)$$

AH is parallel to BG. (?)

Similarly, prove AF and DG equal and parallel. Q.E.D.

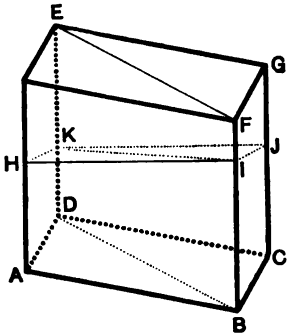
919. Corollary. *Any two opposite faces of a parallelopiped may be taken as its bases.*

920. Exercise. **Three concurrent edges of a rectangular parallelopiped are 3, 4, and 6 ft. Find its surface.**

921. Exercise. **Show that the face diagonals DE and FC (see figure, § 918) are parallel.**

Proposition VI. Theorem

922. *The plane passed through two diagonally opposite edges of a parallelopiped divides the parallelopiped into two equivalent triangular prisms.*



Let the plane *BFED* pass through the diagonally opposite edges of the parallelopiped *AG*.

To Prove that the triangular prisms *ABD–F* and *CBD–F* are equivalent.

Proof. Let *HIJK* be a right section of the parallelopiped.

Faces *AE* and *BG* are parallel. (?)

HK is ∥ to *IJ*. (?)

Similarly, *HI* and *KJ* are parallel.

HIJK is a parallelogram, and $\triangle HKI = \triangle KIJ$. (?)

The prism *ABD–F* is equivalent to a right prism having $\triangle HKI$ for a base and *BF* for an altitude. (§ 916.)

The prism *CBD–F* is equivalent to a right prism having $\triangle KIJ$ for a base and *BF* for an altitude.

These two right prisms have equal bases and the same altitude. They are equal by § 913.

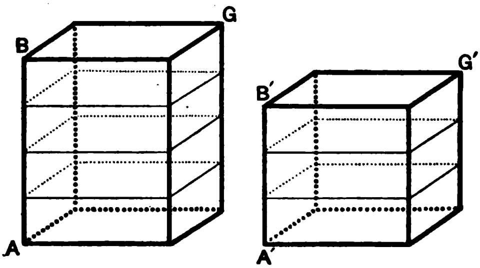
The prisms *ABD–F* and *CBD–F* that are equivalent to these right prisms are also equivalent prisms. Q.E.D.

923. Exercise. The diagonals of a parallelopiped bisect each other.

PROPOSITION VII. THEOREM

924. ***Two rectangular parallelopipeds having equal bases are to each other as their altitudes.***

CASE I. When the altitudes are commensurable.



Let AG and $A'G'$ be two rectangular parallelopipeds having equal bases and commensurable altitudes.

To Prove $$\frac{AG}{A'G'}=\frac{AB}{A'B'}.$$

Proof. Suppose the common unit of measure of the altitudes is contained in AB m times and in $A'B'$ n times.

Then $$\frac{AB}{A'B'}=\frac{m}{n}.$$

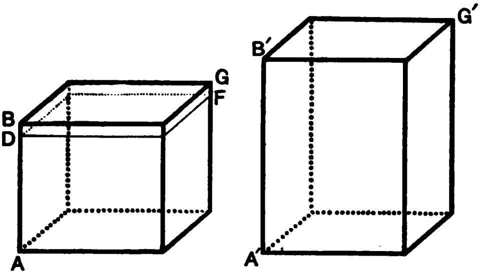
By passing planes through the points of division of the altitudes and parallel to the bases, AG may be divided into m rectangular parallelopipeds, and $A'G'$ into n rectangular parallelopipeds.

The parallelopipeds thus formed are equal. (?)

Then $$\frac{AG}{A'G'}=\frac{m}{n}.$$

Whence $$\frac{AG}{A'G'}=\frac{AB}{A'B'}.$$ Q.E.D

CASE II. When the altitudes are incommensurable.



Let AG and $A'G'$ be two rectangular parallelopipeds having equal bases and incommensurable altitudes.

To Prove $$\frac{AG}{A'G'} = \frac{AB}{A'B'}.$$

Proof. Let $A'B'$ be divided into a number of equal parts, and one of these be applied to AB as a unit of measure.

AB will not contain this unit of measure exactly, but a certain number of these parts will extend from A to D, leaving a remainder DB less than the unit of measure.

Pass DF parallel to the bases.

$$\frac{AF}{A'G'} = \frac{AD}{A'B'}. \quad (?)$$

By increasing indefinitely the number of equal parts into which $A'B'$ is divided the divisions will become smaller and smaller, and the remainder DB will also diminish indefinitely.

$\frac{AF}{A'G'}$ and $\frac{AD}{A'B'}$ are variables, and they are always equal to each other. (?)

The limit of $\frac{AF}{A'G'}$ is $\frac{AG}{A'G'}$ and the limit of $\frac{AD}{A'B'}$ is $\frac{AB}{A'B'}$. (?)

$$\therefore \frac{AG}{A'G'} = \frac{AB}{A'B'}.$$ Q.E.D.

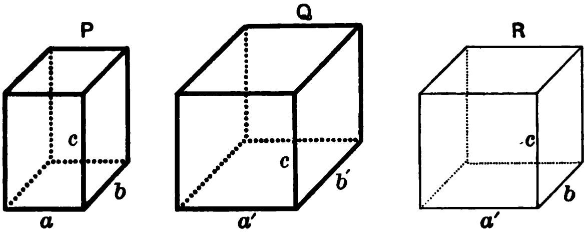
925. Definition. The *dimensions* of a rectangular parallelopiped are the three edges that meet at a common vertex.

926. Corollary. *Two rectangular parallelopipeds that have two dimensions of one equal respectively to two dimensions of the other, are to each other as their third dimensions.*

927. Exercise. The volumes of two rectangular parallelopipeds having equal bases are a cu. ft. and b cu. ft. respectively. The altitude of the first is c ft. What is the altitude of the second?

Proposition VIII. Theorem

928. *Two rectangular parallelopipeds having equal altitudes are to each other as their bases.*



Let P and Q be two rectangular parallelopipeds having the same altitude, c.

To Prove $$\frac{P}{Q}=\frac{a\times b}{a'\times b'}.$$

Proof. Let R be a third rectangular parallelopiped whose altitude is c and the dimensions of whose base are a' and b.

$$\frac{P}{R}=\frac{a}{a'}. \quad (?) \quad (1) \qquad\qquad \frac{R}{Q}=\frac{b}{b'}. \quad (?) \quad (2)$$

Multiplying (1) and (2) together, member by member,

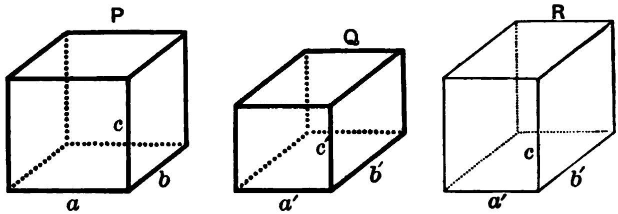
$$\frac{P}{Q}=\frac{a\times b}{a'\times b'}.$$ Q.E.D.

929. COROLLARY. *Two rectangular parallelopipeds that have a dimension of one equal to a dimension of the other, are to each other as the products of their other two dimensions.*

930. EXERCISE. The bases of two rectangular parallelopipeds that have equal altitudes are 9 sq. ft. and 12 sq. ft. respectively. The volume of the first is 96 cu. ft. What is the volume of the second?

PROPOSITION IX. THEOREM

931. *Two rectangular parallelopipeds are to each other as the products of their three dimensions.*



Let P and Q be any two rectangular parallelopipeds.

To Prove $\frac{P}{Q} = \frac{a \times b \times c}{a' \times b' \times c'}$.

Proof. Let R be a third rectangular parallelopiped, having the dimensions a', b', and c.

$$\frac{P}{R} = \frac{a \times b}{a' \times b'}. \quad (?)$$

$$\frac{R}{Q} = \frac{c}{c'}. \quad (?)$$

$$\frac{P}{Q} = \frac{a \times b \times c}{a' \times b' \times c'}. \quad (?)$$ Q.E.D.

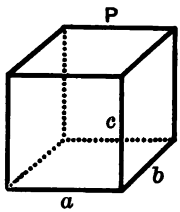
932. EXERCISE. The dimensions of a rectangular parallelopiped are 3 ft., 4 ft., and 5 ft. The dimensions of a second rectangular parallelopiped are 4 ft., 5 ft., and 6 ft. How do their volumes compare?

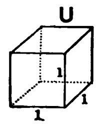
933. Exercise. The dimensions of the rectangular parallelopiped M are each double the corresponding dimensions of the rectangular parallelopiped N. Compare their volumes.

Proposition X. Theorem

934. *The volume of a rectangular parallelopiped is equal to the product of its three dimensions.*

Let P be any rectangular parallelopiped.





To Prove $P = a \times b \times c$.

Proof. Let the cube U, each of whose dimensions is a linear unit, be the unit of measure for volumes.

$$\frac{P}{U} = \frac{a \times b \times c}{1 \times 1 \times 1}. \quad (?)$$

Whence $$P = a \times b \times c \times U.$$

Since U, the unit of measure for volume, is expressed by 1, the last equation may be abbreviated into

$$P = a \times b \times c. \qquad \text{Q.E.D.}$$

935. Corollary I. *The volume of a cube is equal to the cube of its edge.*

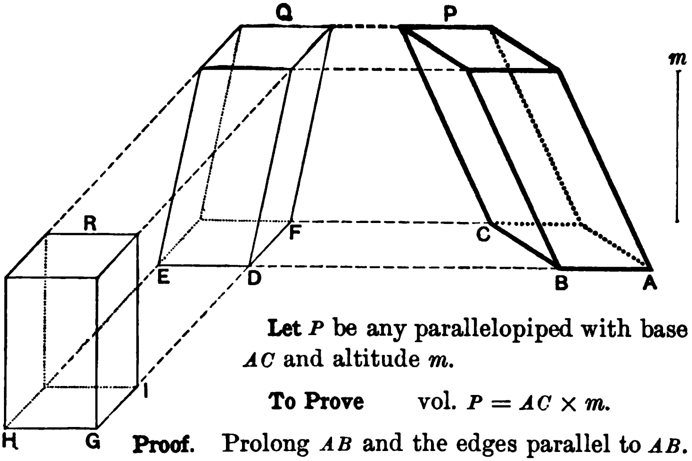
936. Corollary II. *The volume of a rectangular parallelopiped is equal to the product of its base and altitude.*

937. Exercise. The volume of a rectangular parallelopiped is 54 cu. ft. Its base is a square, each side of which is one half the altitude of the parallelopiped. Find the altitude.

938. Exercise. Find the edge of a cube whose volume and entire surface each contain the same number of units.

Proposition XI. Theorem

939. *The volume of any parallelopiped is equal to the product of its base and altitude.*



Let P be any parallelopiped with base AC and altitude m.

To Prove vol. $P = AC \times m$.

Proof. Prolong AB and the edges parallel to AB.

Lay off $DE = AB$.

Pass planes through D and E and $\perp$ to DE, forming Q.
Q is a parallelopiped whose base EF is a rectangle.
Prolong FD and the edges parallel to FD.

Lay off $IG = FD$.

Pass planes through I and G and $\perp$ to IG, forming R.
R is a rectangular parallelopiped. (?)

$$\text{Vol. } R = IH \times m. \quad (?)$$

P, Q, and R are equivalent. (§ 916.)

$$\therefore \text{ vol. } P = IH \times m.$$

The bases of P, Q, and R are equivalent. (?)

$$\therefore \text{ vol. } P = AC \times m.$$

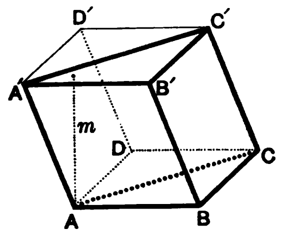
Q.E.D.

940. Exercise. The base of a parallelopiped is a rhombus, one of whose diagonals is equal to its side. The altitude of the parallelopiped is a inches, and is also equal to a side of the base. Find the volume of the parallelopiped.

941. Exercise. The altitude of a parallelopiped is 8 in., and a diagonal of its base divides it into two equilateral triangles each of whose sides is 6 in. Find the volume of the parallelopiped.

Proposition XII. Theorem

942. *The volume of a triangular prism is equal to the product of its base and altitude.*



Let $ABC\text{–}C'$ be any triangular prism having ABC for its base and M for its altitude.

To Prove vol. $ABC\text{–}C' = ABC \times M$.

Proof. Complete the parallelopiped $ABCD\text{–}C'$. (?)

$$\text{Vol. } ABC\text{–}C' = \tfrac{1}{2} \text{ vol. } ABCD\text{–}C'. \quad (?)$$

$$\therefore \text{ vol. } ABC\text{–}C' = \tfrac{1}{2}\, ABCD \times M \quad (?)$$

or $$\text{vol. } ABC\text{–}C' = ABC \times M.$$ Q.E.D.

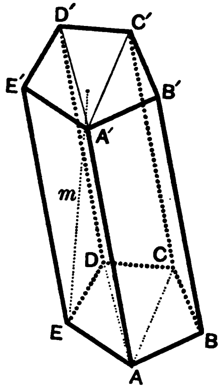
943. Exercise. The altitude of a prism is 8 ft. Its base is a triangle whose sides are 6 ft., 8 ft., and 10 ft., respectively. What is the volume of the prism?

944. Exercise. The volume of a triangular prism is a cu. in. Its altitude is b in., and its base is an equilateral triangle. Find a side of the base.

945. Exercise. The volume of any triangular prism is equal to half the product of any lateral face by the perpendicular to this face from any point of the opposite edge.

Proposition XIII. Theorem

946. *The volume of any prism is equal to the product of its base and altitude.*



Let $ABCDE$–C' be any prism, $ABCDE$ its base, and m its altitude.

To Prove

vol. $ABCDE$–$C' = ABCDE \times m$.

Proof. Pass planes through AA' and the edges parallel to AA'. The prism is now divided into a number of triangular prisms having a common altitude m.

The volume of each triangular prism is the product of its base and the altitude m. (?)

The sum of the volumes of the triangular prisms, or the volume of the prism $ABCDE$–C', is equal to the sum of the triangular bases times the altitude m.

The sum of these triangular bases $= ABCDE$.

$\therefore$ vol. $ABCDE$–$C' = ABCDE \times m$. Q.E.D.

947. Corollary I. *Prisms having equivalent bases and equal altitudes are equivalent.*

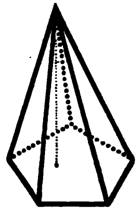
948. Corollary II. *Prisms are to each other as the products of their bases and altitudes. If their bases are equivalent, they are to each other as their altitudes; and if their altitudes are equal, they are to each other as their bases.*

949. Exercise. The altitude of a prism is 10 in., and its base is a regular hexagon, each side of which is 6 in. Find the volume of the prism.

950. Exercise. The volume of a prism is 80 cu. ft. Its altitude is 5 ft. Find the perimeter of its square base.

951. Exercise. The altitude of a prism is *a* in. and its base is *b* sq. in. The altitude of an equivalent prism is *c* in. Find the side of the equilateral triangle that forms its base.

952. Definitions. A *pyramid* is a polyhedron bounded by a polygon, called its base, and a number of triangles having a common vertex.



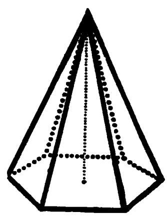
These triangles are called the *lateral faces* of the pyramid, and their point of meeting its *vertex.*

The sum of the lateral faces is the *lateral area* of the pyramid.

The perpendicular distance from the vertex to the base is the *altitude* of the pyramid.

A pyramid is *triangular*, *quadrangular*, etc., according as its base is a triangle, quadrilateral, etc. A triangular pyramid is called a *tetrahedron.*

A *regular pyramid* has a regular polygon for its base, and the altitude of the pyramid meets the base at its center.



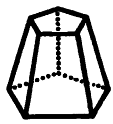
The lateral edges of a regular pyramid are equal. (§ 816.)

The lateral triangles of a regular pyramid are equal and isosceles. (?)

The *slant height* of a regular pyramid is the altitude of one of its lateral faces.

That part of a pyramid between its base and a plane cutting all its lateral edges is a *truncated pyramid.*

If the bases of a truncated pyramid are parallel, it is called a *frustum* of a pyramid.

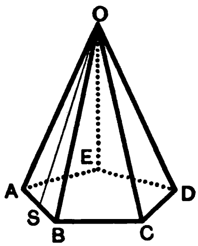


The *altitude* of a frustum of a pyramid is the perpendicular distance between its bases.

The lateral faces of a frustum of a regular pyramid are equal trapezoids (?). The altitude of one of these trapezoids is the *slant height* of the frustum.

PROPOSITION XIV. THEOREM

953. ***The lateral area of a regular pyramid is equal to one half the product of its slant height by the perimeter of its base.***



Let *O–ABCDE* be a regular pyramid and let *OS* be its slant height.

To Prove lateral area *O–ABCDE* $= \frac{1}{2} OS \times$ perimeter *ABCDE.*

Proof. The area of each triangle of the lateral surface is equal to the product of its base and $\frac{1}{2}$ *OS.* (?)

Since the sum of these triangles makes the lateral area of the pyramid, and the sum of their bases the perimeter of its base,

∴ lateral area *O–ABCDE* $= \frac{1}{2} OS \times$ per. *ABCDE.* Q.E.D.

954. COROLLARY. *The lateral area of the frustum of a regular pyramid is equal to the product of its slant height and one half the sum of the perimeters of its bases.*

955. EXERCISE. The slant height of a regular hexagonal pyramid is 10 ft. Each side of its base is 8 ft. What is its lateral area?

956. EXERCISE. The altitude of a regular quadrangular pyramid is 4 in. One side of its base is 6 in. Find its lateral area.

957. EXERCISE. Find the lateral area of the frustum formed by a plane bisecting the altitude of the pyramid of § 956.

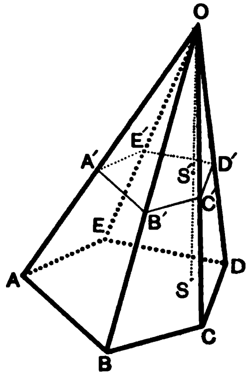
Proposition XV. Theorem

958. *If a pyramid is cut by a plane parallel to the base,*

I. *The altitude and the lateral edges are divided proportionally.*

II. *The section is a polygon similar to the base.*

Let $O\text{–}ABCDE$ be cut by the plane $A'D'$ parallel to the base AD.



I. **To Prove** $\dfrac{OS}{OS'}=\dfrac{OA}{OA'}=\dfrac{OB}{OB'}$, etc.

Proof. Pass a plane through O parallel to the base. Apply § 849.

II. **To Prove** $ABCDE$ and $A'B'C'D'E'$ similar.

Proof. Show that they are mutually equiangular, and that their corresponding sides are proportional.

959. Corollary I. *Parallel sections of a pyramid are to each other as the squares of their distances from the vertex.*

$$\frac{ABCDE}{A'B'C'D'E'}=\frac{\overline{AB}^2}{\overline{A'B'}^2}\ (?)\qquad \frac{\overline{AB}^2}{\overline{A'B'}^2}=\frac{\overline{OB}^2}{\overline{OB'}^2}\ (?)$$

$$\frac{\overline{OB}^2}{\overline{OB'}^2}=\frac{\overline{OS}^2}{\overline{OS'}^2}\ (?)\qquad \therefore\ \frac{ABCDE}{A'B'C'D'E'}=\frac{\overline{OS}^2}{\overline{OS'}^2}.$$

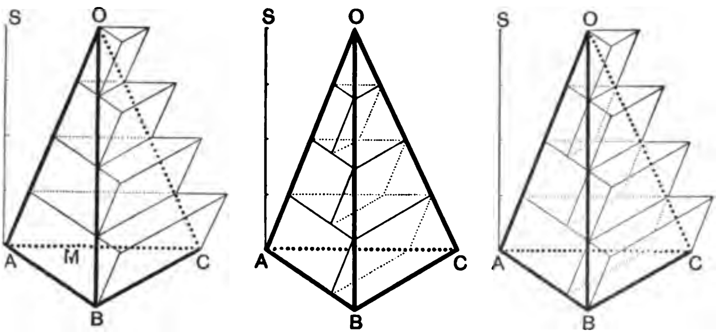
960. Corollary II. *If two pyramids have equal altitudes, sections parallel to their bases and equally distant from their vertices have the same ratio as the bases.* [Apply § 959.]

961. Corollary III. *If two pyramids have equal altitudes and equivalent bases, sections parallel to their bases and equally distant from their vertices are equivalent.*

962. Exercise. The base of a pyramid is 10 sq. in. and a plane parallel to the base cuts the altitude 2 in. from the vertex. If the altitude of the pyramid is 8 in., what is the area of the section made by the plane parallel to the base?

Proposition XVI. Theorem

963. *The volume of a triangular pyramid is the limit of the sum of the volumes of a series of inscribed or circumscribed prisms of equal altitude, if the number of prisms is indefinitely increased.*



Let O–ABC be any triangular pyramid, and AS be its altitude.

To Prove that the volume of O–ABC is the limit of the sum of the volumes of a series of inscribed or circumscribed prisms of equal altitude, if the number of prisms is indefinitely increased.

Proof. Divide the altitude AS into a number of equal parts, and through the points of division pass planes parallel to the base, forming triangular sections.

Using ABC and the triangular sections as lower bases, construct prisms whose lateral edges shall be parallel to AO, and whose altitudes shall be the distance between the parallel sections. These prisms may be said to be *circumscribed* about the pyramid $O\text{-}ABC$.

Also using the triangular sections as upper bases, construct prisms whose lateral edges shall be parallel to AO, and whose altitudes shall be the distance between the parallel sections. This set of prisms may be said to be *inscribed* in the pyramid $O\text{-}ABC$.

For every circumscribed prism there is an equivalent inscribed prism, except for the circumscribed prism having ABC for its lower base, for which there is no equivalent inscribed prism.

The difference between the sum of the circumscribed prisms and the sum of the inscribed prisms is prism M.

By increasing the number of divisions into which AS is divided, the divisions can be made as small as we please, and the volume of prism M can be made as small as we please, although not equal to zero.

The difference between the sum of the circumscribed prisms and the sum of the inscribed prisms can therefore be made as small as we please, but not equal to zero.

The volume of the pyramid $O\text{-}ABC$ differs from the sum of the circumscribed prisms or the sum of the inscribed prisms by less than they differ from each other.

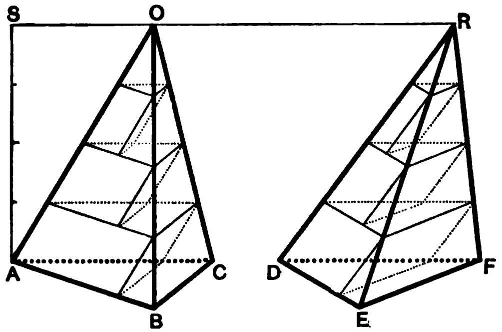
Consequently the difference between the volume of the pyramid and either the sum of the circumscribed prisms or the sum of the inscribed prisms can be made less than any assignable quantity, but not equal to zero.

Therefore the volume of the pyramid is the limit of the sum of the circumscribed prisms or of the inscribed prisms as their number is indefinitely increased. Q.E.D.

Proposition XVII. Theorem

964. *Triangular pyramids having equal altitudes and equivalent bases are equivalent.*

Let the pyramids *O–ABC* and *R–DEF* have equivalent bases and a common altitude *AS*.



To Prove *O–ABC* and *R–DEF* equivalent.

Proof. Divide the altitude *AS* into a number of equal parts, and through the points of division pass planes parallel to the plane of the bases.

The corresponding sections made by these parallel planes are equivalent. (?)

Inscribe in each pyramid a series of prisms having the triangular sections as upper bases, and the distance between the sections as their altitudes.

The corresponding prisms of the two pyramids are equivalent. (?)

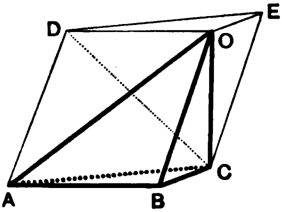
The sum of the prisms inscribed in *O–ABC* is equivalent to the sum of the prisms inscribed in *R–DEF*.

If the number of divisions into which *AS* is divided is indefinitely increased, the sum of the prisms inscribed in *O–ABC* approaches the volume of *O–ABC* as its limit, and the sum of the prisms inscribed in *R–DEF* approaches the volume of *R–DEF* as its limit. (?)

Since these variable sums are always equal, their limits are equal. Consequently vol. *O–ABC* = vol. *R–DEF*. (?) Q.E.D.

Proposition XVIII. Theorem

965. *The volume of a triangular pyramid is equal to one third the product of its base and altitude.*



Let $O\text{–}ABC$ be any triangular pyramid.

To Prove the volume of $O\text{–}ABC = \frac{1}{3}$ the product of its base and altitude.

Proof. Construct on the base ABC the triangular prism BD, with its lateral edges AD and CE each equal and parallel to OB.

The prism BD is made up of the triangular pyramid $O\text{–}ABC$ and the quadrangular pyramid $O\text{–}ACED$.

Pass a plane through OC and OD, dividing the quadrangular pyramid $O\text{–}ACED$ into two triangular pyramids $O\text{–}ACD$ and $O\text{–}CED$.

Pyramids $O\text{–}ACD$ and $O\text{–}CED$ are equivalent. (?)

Pyramid $O\text{–}CED$ may be read $C\text{–}ODE$.

Pyramids $C\text{–}ODE$ and $O\text{–}ABC$ are equivalent.

∴ the three triangular pyramids composing the prism BD are equivalent.

$O\text{–}ABC$ is equal to one third of the prism BD.

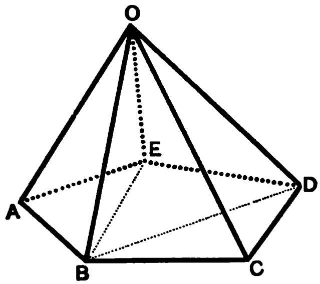
$O\text{–}ABC = \frac{1}{3}$ the product of its base and altitude. (?) Q.E.D.

966. Exercise. Find the altitude of a triangular pyramid whose volume is 50 cu. in. and whose base is 12 sq. in.

967. Exercise. The volume of a parallelopiped is m cu. in. Find the altitude of an equivalent pyramid whose base is one of the triangles into which the base of the parallelopiped is divided by its diagonals.

PROPOSITION XIX. THEOREM

968. *The volume of any pyramid is equal to one third the product of its base and altitude.*



Let $O\text{-}ABCDE$ be any pyramid.

To Prove the volume of $O\text{-}ABCDE = \frac{1}{3}$ the product of its base and altitude.

Suggestion. Divide the pyramid into triangular pyramids and apply § 965.

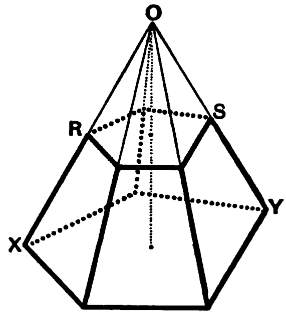
969. COROLLARY I. *The volumes of pyramids are to each other as the products of their bases and altitudes; if their bases are equivalent, the pyramids are to each other as their altitudes; and if their altitudes are equal, the pyramids are to each other as their bases.*

970. COROLLARY II. *The volume of any polyhedron may be found by dividing it up into triangular pyramids, computing their volumes separately, and taking the sum of their volumes.*

971. EXERCISE. The altitude of a pyramid is 8 ft. and its base is a regular pentagon each side of which is 4 ft. Find the volume of the pyramid. [§ 754.]

PROPOSITION XX. THEOREM

972. *The volume of the frustum of a pyramid is equal to the sum of its bases and a mean proportional between its bases, multiplied by one third of its altitude.*



Let XS be a frustum of the pyramid O–XY; and let B represent the area of the lower base, b the area of the upper base, and a the altitude of the frustum.

To Prove

vol. $XS = \frac{1}{3}a(B + b + \sqrt{B \times b})$.

Proof. Let m represent the altitude of the pyramid O–RS, and v represent the volume of the frustum XS.

The volume of the frustum XS is equal to the difference between the volumes of the pyramids O–XY and O–RS.

$$v = \tfrac{1}{3}B(a + m) - \tfrac{1}{3}b \times m. \quad (?)$$

(1) $$v = \tfrac{1}{3}B \times a + \tfrac{1}{3}(B - b)m. \quad (?)$$

$$\frac{B}{b} = \frac{(a + m)^2}{m^2}. \quad (?)$$

$$\frac{\sqrt{B}}{\sqrt{b}} = \frac{a + m}{m}, \text{ whence } m = \frac{a\sqrt{b}}{\sqrt{B} - \sqrt{b}}.$$

Substitute this value of m in (1).

$$v = \tfrac{1}{3}B \times a + \tfrac{1}{3}(\sqrt{B} + \sqrt{b})a\sqrt{b}.$$

$$v = \tfrac{1}{3}B \times a + \tfrac{1}{3}a\sqrt{B \times b} + \tfrac{1}{3}b \times a.$$

$$v = \tfrac{1}{3}a(B + b + \sqrt{B \times b}).$$ Q.E.D.

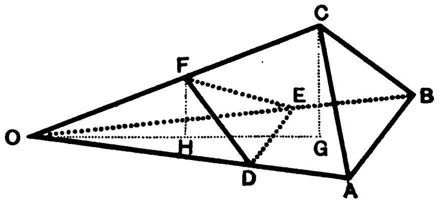
973. COROLLARY. *The frustum of a pyramid is equivalent to the sum of three pyramids having a common altitude equal to that of the frustum, and whose bases are the upper and lower bases of the frustum and a mean proportional between them.*

974. Exercise. The upper and lower bases of the frustum of a pyramid contain 32 sq. ft. and 50 sq. ft. respectively. The altitude of the frustum is 12 ft. Find its volume.

975. Exercise. The slant height of the frustum of a regular pyramid is 16 ft.; the sides of its square bases 30 ft. and 12 ft. Find its volume.

Proposition XXI. Theorem

976. *Two triangular pyramids that have a trihedral angle of one equal to a trihedral angle of the other are to each other as the products of the edges including the equal angles.*



Let the triangular pyramids $O\text{–}ABC$ and $O\text{–}DEF$ have the trihedral angle O in common.

To Prove $$\frac{O\text{–}ABC}{O\text{–}DEF}=\frac{OA\times OB\times OC}{OD\times OE\times OF}.$$

Proof. Draw CG and $FH \perp$ to face OBA.

$$\frac{O\text{–}ABC}{O\text{–}DEF}=\frac{OAB\times CG}{ODE\times FH}. \quad (?)$$

$$\frac{OAB}{ODE}=\frac{OA\times OB}{OD\times OE}. \quad (\S\ 613)$$

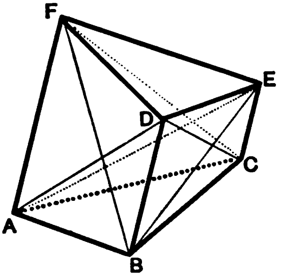
$$\frac{CG}{FH}=\frac{OC}{OF}. \quad (?)$$

$$\therefore \frac{O\text{–}ABC}{O\text{–}DEF}=\frac{OA\times OB\times OC}{OD\times OE\times OF}.$$ Q.E.D.

977. Exercise. If two triangular pyramids have a common trihedral angle, and one of the faces about this angle in each equivalent, the pyramids are to each other as the edges of the common trihedral angle that lie opposite the equivalent faces.

Proposition XXII. Theorem

978. *A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism and whose vertices are the three vertices of the inclined section.*



Let $ABC\text{–}DEF$ be a truncated triangular prism.

To Prove $ABC\text{–}DEF = D\text{–}ABC + E\text{–}ABC + F\text{–}ABC.$

Proof. Pass the planes DAC and DFC, dividing the truncated prism into three pyramids, $D\text{–}ABC$, $D\text{–}ACF$, and $D\text{–}CEF$.

$$D\text{–}ACF = B\text{–}ACF. \quad (?)$$

$B\text{–}ACF$ may be read $F\text{–}ABC$.

$$D\text{–}CEF = B\text{–}CEF. \quad (?)$$

$$\triangle ACE = \triangle CEF. \quad (?)$$

$$\therefore B\text{–}CEF = B\text{–}ACE.$$

$B\text{–}ACE$ may be read $E\text{–}ABC$.

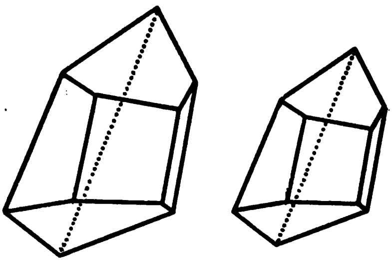
$$\therefore ABC\text{–}DEF = D\text{–}ABC + E\text{–}ABC + F\text{–}ABC. \qquad \text{Q.E.D.}$$

979. Corollary I. *The volume of a truncated right triangular prism is equal to the product of its base and one third the sum of its lateral edges.*

980. Corollary II. *The volume of any truncated triangular prism is equal to the product of its right section by one third the sum of its lateral edges.*

[The right section divides the truncated prism into two right truncated prisms. Apply § 979.]

981. Definitions. Two polyhedrons are *similar* if they have the same number of faces similar each to each and similarly placed, and have their corresponding polyhedral angles equal.



The faces, angles, edges, and lines that are similarly placed in the two polyhedrons are called *homologous faces, angles, edges,* and *lines.*

By proofs analogous to those of the corresponding propositions of plane geometry, the following principles may be deduced:

In similar polyhedrons, homologous lines are proportional.

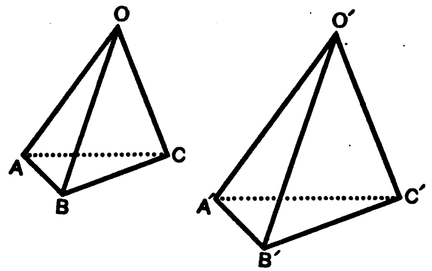
Any two homologous faces of two similar polyhedrons are to each other as the squares of any two homologous lines.

Any two homologous faces of two similar polyhedrons are like parts of the surfaces of the polyhedrons.

The surfaces of two similar polyhedrons are to each other as the squares of any two homologous lines.

Proposition XXIII. Theorem

982. *Two tetrahedrons are similar if three faces of one are similar respectively to three faces of the other, and are similarly placed.*



Let the tetrahedron $O\text{–}ABC$ have the faces AOB, AOC, and BOC similar respectively to the faces $A'O'B'$, $A'O'C'$, and $B'O'C'$ of the tetrahedron $O'\text{–}A'B'C'$.

To Prove $O\text{–}ABC$ and $O'\text{–}A'B'C'$ similar.

Proof. Trihedral $\angle O =$ trihedral $\angle O'$. (?)

$\triangle$s ABC and $A'B'C'$ are similar. (?)

The remaining trihedral angles are equal, each to each. (?)

By definition the tetrahedrons are similar. Q.E.D.

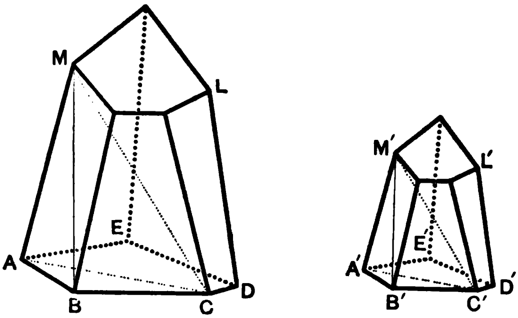
983. Corollary. *Two tetrahedrons are similar if a dihedral angle of one is equal to a dihedral angle of the other, and the faces including the equal angles are similar each to each, and are similarly placed.*

[Use § 900 to show that they have a trihedral angle equal. Then apply § 982.]

984. Exercise. If two tetrahedrons have a face of one similar to a face of the other, and two dihedral angles whose edges form an angle of one of these faces equal respectively to the two dihedral angles whose edges form the homologous angle of the other triangle, and similarly placed, the tetrahedrons are similar.

Proposition XXIV. Theorem

985. ***Two similar polyhedrons can be divided into the same number of tetrahedrons, similar each to each and similarly placed.***



Let $AD-L$ and $A'D'-L'$ be two similar polyhedrons.

To Prove that they can be divided into the same number of tetrahedrons, similar each to each and similarly placed.

Proof. Divide the faces of $AD-L$, except those that have M as a vertex, into triangles.

Divide the faces of $A'D'-L'$, except those that have M' as a vertex, into triangles similar to the triangles of $AD-L$. (?)

Pass planes through M and each side of the triangles of $AD-L$, also through M' and each side of the triangles of $A'D'-L'$.

$AD-L$ and $A'D'-L'$ are now divided into the same number of tetrahedrons, similarly placed.

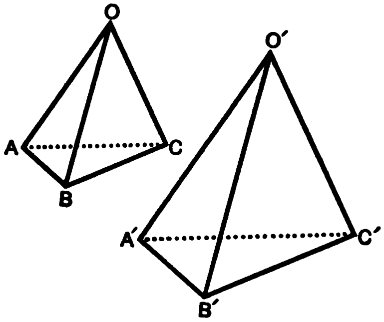
$M-ABC$ and $M'-A'B'C'$ have dihedral $\angle AB =$ dihedral $\angle A'B'$, and ⮝ MAB and ABC are similar to ⮝ $M'A'B'$ and $A'B'C'$ respectively. (?)

$\therefore$ $M-ABC$ and $M'-A'B'C'$ are similar. (?)

In like manner each tetrahedron of $AD-L$ may be proved similar to the homologous tetrahedron of $A'D'-L'$. Q.E.D.

Proposition XXV. Theorem

986. ***Similar tetrahedrons are to each other as the cubes of their homologous edges.***



Let $O\text{–}ABC$ and $O'\text{–}A'B'C'$ be two similar tetrahedrons.

To Prove $$\frac{O\text{–}ABC}{O'\text{–}A'B'C'}=\frac{\overline{OB}^3}{\overline{O'B'}^3}.$$

Proof. $$\frac{O\text{–}ABC}{O'\text{–}A'B'C'}=\frac{OA\times OB\times OC}{O'A'\times O'B'\times O'C'}.\quad (?)$$

$$\frac{OA}{O'A'}=\frac{OB}{O'B'}\ (?)\quad \text{and}\quad \frac{OC}{O'C'}=\frac{OB}{O'B'}.\quad (?)$$

Whence $$\frac{O\text{–}ABC}{O'\text{–}A'B'C'}=\frac{\overline{OB}^3}{\overline{O'B'}^3}.\quad (?)$$ Q.E.D.

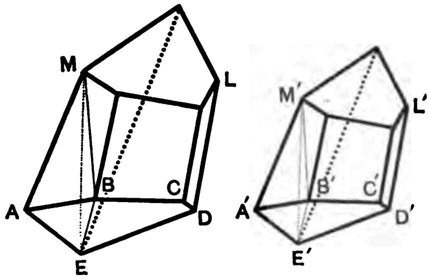
987. Exercise. The homologous edges of two similar tetrahedrons are 3 in. and 4 in. respectively. The volume of the former is 50 cu. in. Find the volume of the other.

988. Exercise. The volume of a given tetrahedron is 40 cu. ft. Construct a similar tetrahedron whose volume shall be 5 cu. ft.

989. Exercise. From a given tetrahedron cut off a frustum whose volume shall equal $\frac{19}{27}$ of the given tetrahedron.

PROPOSITION XXVI. THEOREM

990. ***Similar polyhedrons are to each other as the cubes of their homologous edges.***



Let $AD-L$ and $A'D'-L'$ be two similar polyhedrons.

To Prove $$\frac{AD-L}{A'D'-L'}=\frac{\overline{MA}^3}{\overline{M'A'}^3}.$$

Proof. Divide the polyhedrons into similar tetrahedrons having the common vertices M and M'.

Designate the tetrahedrons of $AD-L$ by T_1, T_2, T_3, T_4, etc., and the tetrahedrons of $A'D'-L'$ by T'_1, T'_2, T'_3, T'_4, etc.

$$\frac{T_1}{T'_1}=\frac{\overline{ME}^3}{\overline{M'E'}^3},\ (?)\qquad \frac{T_2}{T'_2}=\frac{\overline{ME}^3}{\overline{M'E'}^3}.$$

Whence $$\frac{T_1}{T'_1}=\frac{T_2}{T'_2}.$$

Similarly $$\frac{T_2}{T'_2}=\frac{T_3}{T'_3}\ \text{and}\ \frac{T_3}{T'_3}=\frac{T_4}{T'_4},\ \text{etc.}$$

Whence $$\frac{T_1+T_2+T_3,\ \text{etc.}}{T'_1+T'_2+T'_3,\ \text{etc.}}=\frac{T_1}{T'_1},\quad (?)$$

or $$\frac{AD-L}{A'D'-L'}=\frac{T'}{T'_1}.$$

$$\therefore\ \frac{AD-L}{A'D'-L'}=\frac{\overline{MA}^3}{\overline{M'A'}^3}.\quad (?)$$ Q.E.D.

991. Exercise. The volume of a certain polyhedron is 135 cu. yds. Construct a polyhedron similar to the given polyhedron, and having a volume 40 cu. yds. Compare the surface of the constructed polyhedron with that of the given one.

992. Definition. A *regular polyhedron* is a polyhedron whose faces are equal regular polygons, and whose polyhedral angles are equal.

993. The Number of Regular Polyhedrons. Each polyhedral angle has three or more faces, and the sum of its plane angles is less than 4 R.A.'s.

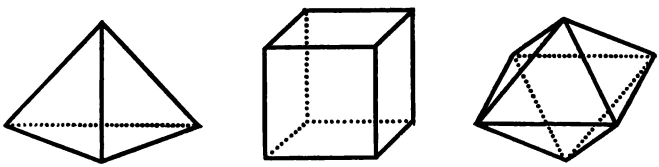
Show that the number of equilateral triangles that can be used to form a polyhedral angle is *three*, *four*, or *five*.

Show that the number of squares that can be used to form a polyhedral angle is *three*.

Show that the number of regular pentagons that can be used to form a polyhedral angle is *three*.

Show that regular polygons having more than five sides cannot be used to form a polyhedral angle.

There are, therefore, five regular polyhedrons. Three of these are bounded by equilateral triangles, one by squares, and one by regular pentagons.



Regular Tetrahedron   Regular Hexahedron   Regular Octahedron

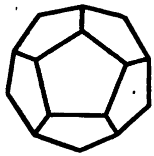
The *regular tetrahedron* is bounded by four equilateral triangles.

The *regular hexahedron* (or cube) is bounded by six squares.

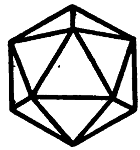
The *regular octahedron* is bounded by eight equilateral triangles.

The *regular dodecahedron* is bounded by twelve regular pentagons.

The *regular icosahedron* is bounded by twenty equilateral triangles.

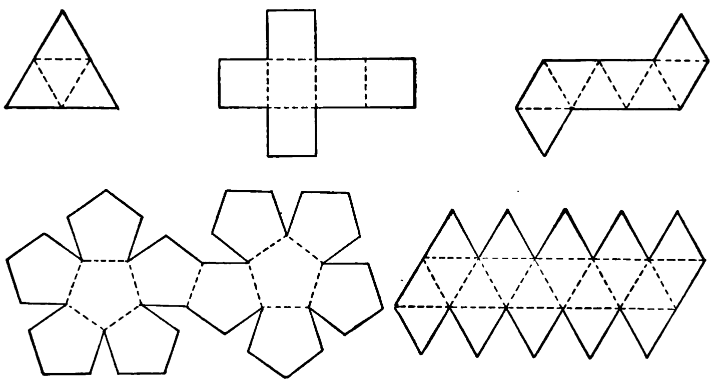


Regular Dodecahedron



Regular Icosahedron

To construct the regular polyhedrons cut out cardboard as indicated in the following diagrams. Fold on the broken lines and bring the edges together.



EXERCISES

1. The lateral surface of a pyramid is greater than its base.

2. In any rectangular parallelopiped the square of a diagonal is equal to the sum of the squares of three edges that meet at a common vertex.

3. The altitude of a pyramid is divided into four equal parts by planes parallel to the base. Find the ratio to one another of the four solids into which the pyramid is divided.

4. The volume of a right prism is 480 cu. ft. Its base is a R.A. triangle whose legs are 16 ft. and 12 ft. Find its lateral area.

5. The diagonal of a cube is equal to the product of its edge by $\sqrt{3}$.

6. The volume of a cube is $2\frac{10}{27}$ cu. in. Find the length of its diagonal.

7. In a tetrahedron planes passed through the three lateral edges and the middle points of the sides of the base pass through a common line.

8. The volume of a regular tetrahedron is equal to the cube of an edge multiplied by $\frac{1}{12}\sqrt{2}$.

9. Find the surface and volume of a regular tetrahedron whose edge is 4 in.

10. The diagonals of a rectangular parallelopiped are equal, and pass through a common point.

11. The lines joining the points of intersection of the diagonals of the opposite faces of a rectangular parallelopiped pass through a common point.

12. If E, F, G, and H are the middle points of the edges AB, AD, CD, and BC respectively of the tetrahedron $ABCD$, prove $EFGH$ a parallelogram.

13. The volume of a regular prism is equal to the product of its lateral area by one half the apothem of its base.

14. The areas of the bases of the frustum of a pyramid are 15 sq. in. and 50 sq. in. The altitude of the frustum is 7 in. Find the altitude of the pyramid.

15. The base of a pyramid is a square, and its lateral faces are equilateral triangles. If its altitude is 6 ft., find the volume and lateral area.

16. The lines joining each vertex of a tetrahedron with the point of intersection of the medial lines of the opposite face all meet in a common point, which divides each line in the ratio 1 : 4. [The point of intersection is the *center of gravity* of the tetrahedron.]

17. In any parallelopiped the sum of the squares of the four diagonals is equal to the sum of the squares of the twelve edges.

18. In a rectangular parallelopiped three of the edges are 8 in., 9 in., and 12 in. respectively. Find the length of a diagonal of the parallelopiped.

19. The diagonal of a cube is a inches. Find its volume.

20. Any line through the point of intersection of the diagonals of a parallelopiped, and terminating in the surface, is bisected at that point.

21. Any plane through the point of intersection of the diagonals of a parallelopiped divides the parallelopiped into two equivalent solids.

22. The sum of two opposite lateral edges of a truncated parallelopiped is equal to the sum of the other two lateral edges.

23. The middle points of the edges of a regular tetrahedron are the vertices of a regular octahedron.

24. What is the edge of a cube whose entire surface is one square foot?

25. In a regular pyramid the sum of the squares of the lateral edges is equal to $\frac{1}{4}$ the sum of the squares of the base edges increased by n times the square of the slant height. [n = no. of sides of base.]

26. A section of a tetrahedron made by a plane parallel to two non-intersecting edges is a parallelogram.

27. If the diagonals of a quadrangular prism pass through a common point, the figure is a parallelopiped.

28. Given the lengths of the diagonals of the three faces about a trihedral angle of a rectangular parallelopiped to determine the edges.

29. The plane that bisects a dihedral angle of a tetrahedron divides the opposite face into two segments that are proportional to the areas of the adjacent faces.

30. The straight lines joining the middle points of the opposite edges of a tetrahedron all pass through the center of gravity of the tetrahedron.

31. If from any point within a regular tetrahedron perpendiculars be drawn to the faces, their sum is equal to an altitude of the tetrahedron.

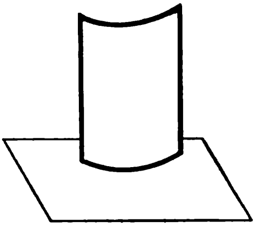
32. On three given lines in space that intersect in a common point, as edges, construct a parallelopiped.

33. If a pyramid is cut by three parallel planes so that the distance of one of the planes from the vertex is a mean proportional between the distances of the other two planes from the vertex, then is the section formed by that plane a mean proportional between the other two sections.

34. Divide a tetrahedron into four equivalent tetrahedrons.

BOOK VIII

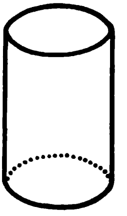
994. Definitions. A *cylindrical surface* is a surface generated by a moving straight line that constantly intersects a fixed curve, and in all its positions is parallel to a fixed straight line not in the plane of the given curve.



The moving line is called the *generatrix;* the fixed curve, the *directrix.*

The generatrix in any of its positions is called an *element* of the cylindrical surface.

If the directrix is a closed convex[1] curve, the solid bounded by a cylindrical surface and two parallel plane surfaces is called a *cylinder.* The cylindrical surface is called its *lateral surface,* and the parallel plane surfaces are its *bases.*



The *altitude* of a cylinder is the perpendicular distance between its bases.

The elements of a cylinder are equal. (?)

A *right cylinder* is a cylinder whose elements are perpendicular to its bases, and the elements of an *oblique cylinder* are oblique to its bases.

A *circular cylinder* is a cylinder whose bases are circles.

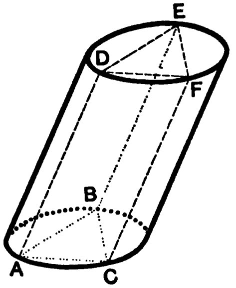
A *cylinder of revolution* is a right circular cylinder, and is generated by revolving a rectangle about one of its sides as an axis.

A *section of a cylinder* is the figure formed by its intersection with a plane passing through it.

[1] A curve is *convex* if a straight line can intersect it in only two points.

PROPOSITION I. THEOREM

995. *The bases of a cylinder are equal.*



Let ABC and DEF be the bases of the cylinder AE.

To Prove ABC and DEF equal.

Proof. Let A, B, and C be any three points in the perimeter of the lower base. From these points draw the elements AD, BE, and CF.

Draw AB, BC, CA, DE, EF, and FD.

Prove $\triangle$ ABC and DEF equal.

The base ABC may be placed on the base DEF so that the points A, B, and C shall coincide with D, E, and F. (?)

But A, B, and C are *any* points in the perimeter of the lower base. Therefore all points in the perimeter of the lower base will coincide with corresponding points of the upper base, and the bases are equal. Q.E.D.

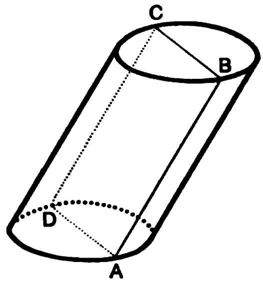
996. COROLLARY I. *A section of a cylinder made by a plane parallel to the base, is equal to the base.*

997. COROLLARY II. *Sections of a cylinder made by parallel planes that cut all the elements are equal.*

998. EXERCISE. Show that a right section of an oblique circular cylinder is not a circle.

Proposition II. Theorem

999. *Any section of a cylinder made by a plane passing through an element is a parallelogram.*



Let *ABCD* be a section of the cylinder *AC* made by a plane passing through the element *AB*.

To Prove *ABCD* a parallelogram.

Proof. Suppose a line drawn through *D* ∥ to *AB*.

This line lies in the plane *ABCD*. (?)

This line is an element of *AC*. (?)

This line is the intersection of the plane *ABCD* with the lateral surface. It is *DC*.

DC is ∥ *AB*, and *BC* ∥ to *AD*. (?)

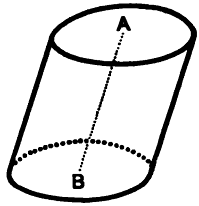
∴ *ABCD* is a parallelogram. Q.E.D.

1000. Corollary. *Any section of a right cylinder made by a plane passing through an element is a rectangle.*

1001. Exercise. Through an element of a right circular cylinder pass a plane cutting off a section whose area is a maximum.

1002. Exercise. Through a point on the lateral surface of a cylinder only one straight line can be drawn lying on the surface.

1003. Definition. The *axis* of a circular cylinder is the straight line joining the centers of its bases.



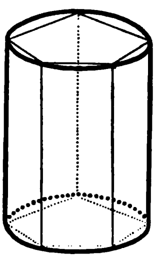
1004. Exercise. The axis of a circular cylinder is parallel to the elements of the lateral surface.

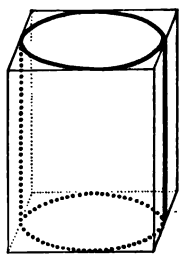
1005. Exercise. The axis of a circular cylinder passes through the centers of all sections of the cylinder that are parallel to the bases.

1006. Definition. A *plane tangent to a cylinder* is a plane that contains one element of the cylinder and no point of the surface without that element.

1007. Exercise. A plane passed through an element of a cylinder and tangent to the base is tangent to the cylinder.

1008. Definitions. A prism is *inscribed in a cylinder* if its bases are inscribed in the bases of the cylinder and its lateral edges are elements of the cylinder.





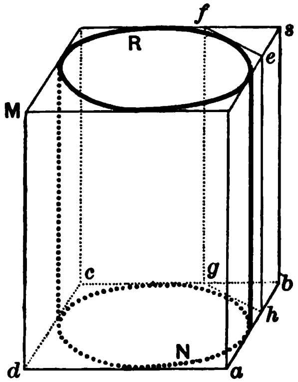
A prism is *circumscribed about a cylinder* if its bases are circumscribed about the bases of the cylinder and its lateral edges are parallel to the elements of the cylinder.

Similar cylinders of revolution are cylinders that are generated by similar rectangles revolved about homologous sides as axes.

1009. Axiom. A plane surface is less than any other surface having the same boundaries.

Proposition III. Theorem

1010. ***The surface of a cylinder is less than the surface of any circumscribed prism.***



Let NR be any cylinder.

To Prove that its surface is less than the surface of any circumscribed prism.

Proof. Of all the surfaces enveloping the solid NR, there must be one whose area is a minimum.

This cannot be a circumscribed prism. For, let $abcd\text{–}s$ be any circumscribed prism. Pass a plane tangent to the lateral surface of the cylinder and intersecting the faces of the prism in eh and fg.

$$efgh < esbh + sfgb + sef + bhg. \qquad (\S\ 1009)$$

∴ the surface of $ahgcd\text{–}M$ is less than that of $abcd\text{–}s$.

The surface of $abcd\text{–}s$ is therefore not the minimum.

The same may be shown of every other surface enveloping NR except the surface of the cylinder.

Therefore, the surface of the cylinder is less than that of any circumscribed prism, or any other surface enveloping the cylinder. Q.E.D.

1011. Corollary. *The surface of a cylinder is greater than the surface of an inscribed prism.* [§ 1009]

1012. Lemma. A convex curve is less than any line that envelopes it and has the same extremities. [Proof similar to that of § 762.]

Proposition IV. Theorem

1013. *If a prism whose base is a regular polygon is inscribed in, or circumscribed about, a circular cylinder, and if the number of sides of the base of the prism be indefinitely increased,*

I. *The volume of the cylinder is the limit of the volume of the prism.*

II. *The perimeter of a right section of the cylinder is the limit of the perimeter of a right section of the prism.*

III. *The lateral area of the cylinder is the limit of the lateral area of the prism.*

Let a prism whose base is a regular polygon be inscribed in the circular cylinder, and one whose base is a similar polygon be circumscribed about the circular cylinder (page 326), and let the number of sides of the base be indefinitely increased.

I. **To Prove** that the volume of the cylinder is the limit of the volume of the prism.

Proof. Designate the volume of the circumscribed prism by V and its base by B. Designate the volume and base of the inscribed prism by v and b respectively. Let the common altitude of the circumscribed and inscribed prisms be designated by a.

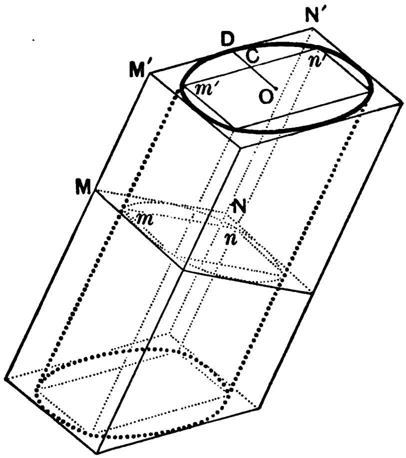
$$V = B \times a \text{ and } v = b \times a. \quad (?)$$

$$V - v = a(B - b).$$

By increasing the number of sides, $B - b$ can be made as small as we please. (?) Since a is constant, $V - v$ can be decreased at pleasure.

But V is always greater than the volume of the cylinder, and v is always less than the volume of the cylinder. (?) Therefore the difference between the volume of the cylinder and the volume of either prism is less than $V - v$, and can consequently be made as small as we please, but never equal to zero.

The volume of the cylinder is therefore the common limit of the volumes of the circumscribed and inscribed prisms, as the number of sides of their bases is indefinitely increased. Q.E.D.



II. **To Prove** that the perimeter of a right section of the cylinder is the limit of the perimeter of a right section of the prism.

Proof. Designate the perimeter of the right section of the circumscribed prism by P, the perimeter of the right section of the inscribed prism by p, and the perimeter of the right section of the cylinder by P'.

The perimeter of the right section is the projection of the perimeter of the base on the plane of the right section.

Prove the inscribed and circumscribed polygons of the right section similar. (§ 880.)

$$\frac{P}{p}=\frac{MN}{mn}=\frac{M'N'}{m'n'}=\frac{OD}{OC}. \quad (?)$$

$$\frac{P-p}{P}=\frac{OD-OC}{OD}. \quad (?) \qquad P-p=\frac{P}{OD}(OD-OC).$$

$OD-OC$ can be made as small as we please, but not equal to zero. (?) Since $\frac{P}{OD}$ does not increase, $P-p$ can be made smaller than any assignable quantity.

But P is always greater than P', and p is always less than P'. (?) Therefore the difference between P' and either P or p is less than $P-p$, and can consequently be made as small as we please, but never equal to zero.

The perimeter of the right section of the cylinder is therefore the limit of the right section of the prism. Q.E.D.

III. **To Prove** that the lateral area of the cylinder is the limit of the lateral area of the prism.

Proof. Let S', S, and s designate the entire surfaces of the cylinder, circumscribed prism, and inscribed prism respectively. Let B and b designate the areas of the bases of the circumscribed prism and inscribed prism respectively, and e a lateral edge of either prism.

$$S=P\times e+2B \text{ and } s=p\times e+2b. \quad (?)$$

$$S-s=e(P-p)+2(B-b).$$

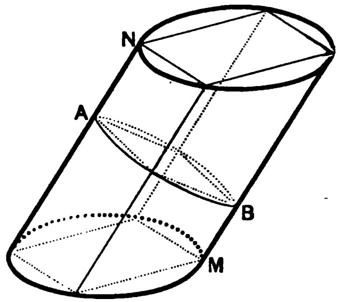
Since $P-p$ and $B-b$ can each be made as small as we please, but not equal to zero (?), and since e and 2 are constants, $S-s$ can be made less than any assignable quantity.

Show that S' is the limit of S and s.

Since the entire surface of the cylinder is the limit of the entire surface of the prism, and the base of the cylinder is the limit of the base of the prism, the lateral area of the cylinder is the limit of the lateral area of the prism. Q.E.D.

PROPOSITION V. THEOREM

1014. *The lateral area of a circular cylinder is equal to the product of the perimeter of a right section of the cylinder by an element.*



Let S denote the lateral area of a circular cylinder, S' denote the lateral area of an inscribed prism whose base is a regular polygon, P denote the perimeter of a right section of the cylinder, and P' the perimeter of the corresponding section of the prism. Let e denote an element of the cylinder.

To Prove $S = P \times e.$

Proof. $S' = P' \times e.$ (?)

As the number of lateral faces of the prism is indefinitely increased, S' approaches S as its limit, and $P' \times e$ approaches $P \times e$ as its limit. Since the members of the equation are two variables that are always equal and each is approaching a limit, their limits are equal.

$$\therefore S = P \times e.$$ Q.E.D.

1015. COROLLARY. *The lateral area of a cylinder of revolution is the product of its altitude by the circumference of its base. If a denotes the altitude and r the radius of the base,*

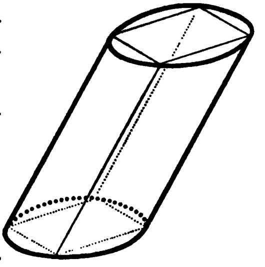
$$S = 2\pi r \times a.$$

1016. EXERCISE. Find the lateral area of a cylinder of revolution whose altitude is 7 ft. and the diameter of whose base is 6 ft.

1017. EXERCISE. The lateral area of a circular cylinder is 60 sq. yd, An element is 8 yd. Find the perimeter of a right section.

PROPOSITION VI. THEOREM

1018. *The volume of a circular cylinder is equal to the product of its base by its altitude.*



Let V denote the volume of the circular cylinder, B its base, and a its altitude. Let V' denote the volume of an inscribed prism, and B' its base (a regular polygon).

To Prove $V = B \times a$.

[Proof similar to that of §1014.]

1019. COROLLARY. *If r is the radius of the base of a circular cylinder, then*

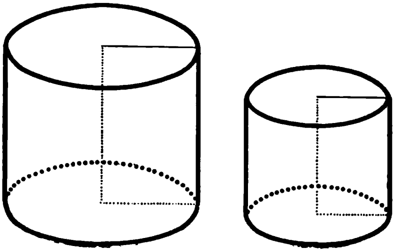
$$V = \pi r^2 \times a.$$

1020. EXERCISE. A circular cylinder contains 100 cu. in. Its altitude is 8 in. What is the radius of its base?

1021. EXERCISE. A cylindrical vessel whose height equals its diameter contains 6283.2 cu. ft. of water. Find its dimensions.

PROPOSITION VII. THEOREM

1022. *The lateral or entire areas of two similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*



Let S and s denote the lateral areas, E and e the entire areas, V and v the volumes, A and a the altitudes, and R and r the radii of the bases of two similar cylinders of revolution.

To Prove $\frac{S}{s}=\frac{E}{e}=\frac{A^2}{a^2}=\frac{R^2}{r^2}$ and $\frac{V^3}{v^3}=\frac{A^3}{a^3}=\frac{R^3}{r^3}$.

Proof. $\frac{S}{s}=\frac{2\pi R\cdot A}{2\pi r\cdot a}=\frac{R\cdot A}{r\cdot a}=\frac{R^2}{r^2}=\frac{A^2}{a^2}$. (?)

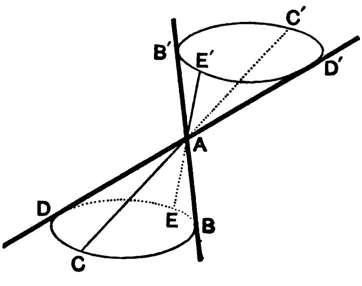
$$\frac{E}{e}=\frac{2\pi R(A+R)}{2\pi r(a+r)}=\frac{R(A+R)}{r(a+r)}=\frac{R^2}{r^2}=\frac{A^2}{a^2}. \quad (?)$$

$$\frac{V}{v}=\frac{\pi R^2\cdot A}{\pi r^2\cdot a}=\frac{R^2\cdot A}{r^2\cdot a}=\frac{R^3}{r^3}=\frac{A^3}{a^3}. \quad (?)$$

Q.E.D.

1023. Exercise. The altitude of one of two similar cylinders of revolution is three times that of the other. Compare their areas and their volumes.

1024. Definitions. A *conical surface* is a curved surface generated by a line moving so that it touches a given curve and passes through a fixed point not in the plane of the curve.



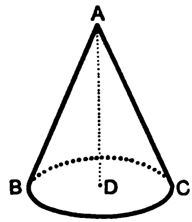
If the straight line AB moves so as to touch the curve $BCDE$ and to pass through the point A, it generates the conical surface A–$BCDE$.

The moving line is called the *generatrix*, the curve is the *directrix*, and the fixed point the *vertex*.

If the generatrix is of indefinite length, the surface generated consists of two portions lying on opposite sides of the vertex and called *nappes*.

The generatrix in any of its positions is called an *element*.

A *cone* is a solid bounded by a conical surface and a plane that cuts all of its elements.



The plane is the *base* of the cone, and the conical surface is the *lateral surface* of the cone.

The *altitude* of a cone is the perpendicular distance from the vertex to the base.

A *circular cone* is a cone whose base is a circle.

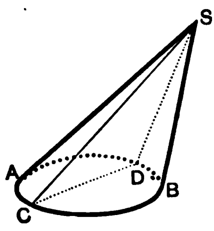
The *axis* of a circular cone is a line from the vertex to the center of the base.

A *right circular cone* is a circular cone in which the axis is perpendicular to the base. It is also called a *cone of revolution,* as it may be generated by revolving a right-angled triangle about one of its legs as an axis.

The hypotenuse of the right-angled triangle in any position is an element of the surface, and is called the *slant height* of the cone.

Proposition VIII. Theorem

1025. *Every section of a cone made by a plane passing through its vertex is a triangle.*



Let *S–ACB* be a cone cut by a plane *SCD* passing through its vertex and cutting the base in the straight line *CD.*

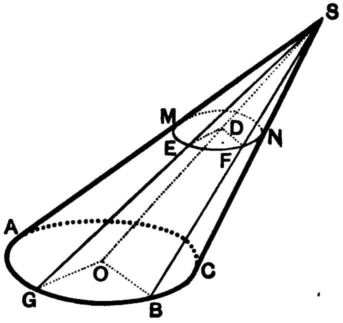
To Prove *SCD* a triangle.

Proof. The straight lines joining *S* with *C* and with *D* are elements and lie in the lateral surface. They also lie in the plane *SCD.* They are the intersections of the plane with the lateral surface of the cone. Since *CD* is a straight line, *SCD* is a triangle. Q.E.D.

1026. Exercise. Every section of a cone of revolution made by a plane passing through its vertex is an isosceles triangle.

PROPOSITION IX. THEOREM

1027. ***A section of a circular cone made by a plane parallel to the base is a circle.***



Let MN be a section of the circular cone $S\text{–}ABC$ parallel to the base.

To Prove MN a circle.

Proof. Draw the axis SO cutting MN at D. Let E and F be any two points in the perimeter of MN. Draw the elements SG and SB passing through E and F respectively. Pass the planes SOG and SOB.

Prove $$\frac{DE}{DF}=\frac{OG}{OB}.$$

Since $OG = OB,\ DE = DF.$

Prove MN a circle. Q.E.D.

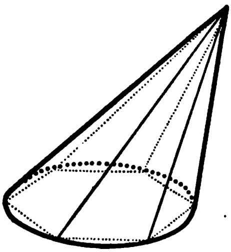
1028. COROLLARY. *The axis of a circular cone passes through the centers of all sections parallel to the base.*

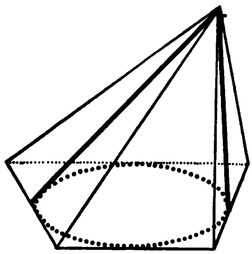
1029. EXERCISE. The area of the section MN is to the area of the base ABC as $\overline{SD}^2$ is to $\overline{SO}^2$.

1030. DEFINITION. A *plane tangent to a cone* is a plane that contains one element of the cone and no point of the cone without that element.

1031. Exercise. A plane passing through an element of a circular cone, and containing a line tangent to the base of the cone at the extremity of the element, is tangent to the cone.

1032. Definitions. A pyramid *is inscribed in a cone* if its base is inscribed in the base of the cone and its vertex coincides with the vertex of the cone.



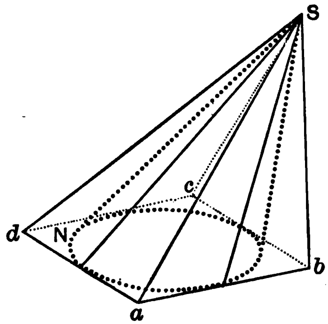


A pyramid is *circumscribed about a cone* if its base is circumscribed about the base of the cone and its vertex coincides with the vertex of the cone.

Proposition X. Theorem

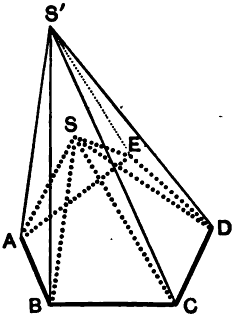
1033. *The surface of a cone is less than the surface of a circumscribed pyramid.*

[Proof similar to that of § 1010.]



1034. Corollary I. *The surface of a cone is greater than the surface of an inscribed pyramid.*

1035. Corollary II. *The surface of a pyramid is less than the surface of a pyramid that envelopes it and has the same base.*



[Produce the plane of one of the lateral faces of the inner pyramid until it cuts the surface of the outer pyramid, forming a new polyhedron whose surface is less than that of the outer pyramid. Produce the plane of the next lateral face of the inner pyramid until it cuts the surface of this polyhedron, forming a second polyhedron whose surface is less than that of the first polyhedron.

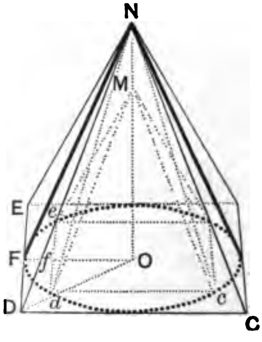
Continue in this way, and the last polyhedron will be the inner pyramid.]

Proposition XI. Theorem

1036. *If a pyramid whose base is a regular polygon is inscribed in or circumscribed about a circular cone, and if the number of sides of the base of the pyramid is indefinitely increased,*

I. *The volume of the cone is the limit of the volume of the pyramid.*

II. *The lateral area of the cone is the limit of the lateral area of the pyramid.*



Let a pyramid whose base is a regular polygon be inscribed in the circular cone, and one whose base is a similar polygon be circumscribed about the circular cone, and let the number of sides of the base be indefinitely increased.

I. **To Prove** that the volume of the cone is the limit of the volume of the pyramid.

[Proof similar to that of § 1013, I.]

II. **To Prove** that the lateral area of the cone is the limit of the lateral area of the pyramid.

Proof. Let S', S, and s designate the entire surfaces of cone, circumscribed pyramid, and inscribed pyramid, respectively. Let B and b designate the bases of the circumscribed pyramid and inscribed pyramid respectively.

Draw dM in the plane of the $\triangle DON$ parallel to DN. Connect M with the remaining vertices of the inscribed base $cde\cdots$, forming a third pyramid $M\text{-}cde\cdots$. Designate the entire surface and base of this pyramid by s' and b' respectively.

Show that Mc, Me, etc., are parallel to NC, NE, etc., respectively.

Show that $\triangle$s Med, Mdc, etc., are similar to $\triangle$s NED, NDC, etc., respectively.

$$\frac{S}{s'}=\frac{\overline{OF}^2}{\overline{Of}^2}.\quad (?) \qquad\qquad \frac{S-s'}{S}=\frac{\overline{OF}^2-\overline{Of}^2}{\overline{OF}^2}.\quad (?)$$

$$S-s'=\frac{S}{\overline{OF}^2}(\overline{OF}^2-\overline{Of}^2).$$

Show that $S-s'$ can be made as small as we please, but not equal to zero.

But s is greater than s' and less than S. (?)

$\therefore$ $S-s$ can be made as small as we please.

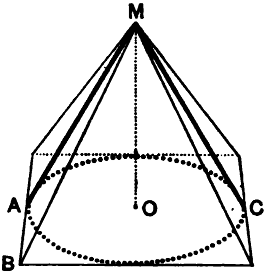
But S' is greater than s and less than S. (?)

Therefore $S-S'$ or $S'-s$ can be made as small as we please, but not equal to zero.

The entire surface of the cone is consequently the limit of the entire surface of the inscribed or circumscribed pyramid. Since the base of the cone is the limit of the base of the pyramid (?), the lateral area of the cone is the limit of the lateral area of the pyramid. Q.E.D.

PROPOSITION XII. THEOREM

1037. *The lateral area of a cone of revolution is equal to one half the product of the circumference of its base by its slant height.*



[Circumscribe about the cone a pyramid having for its base a regular polygon, and proceed as in § 1014.]

1038. COROLLARY. *If S stands for the lateral area, r for the radius of the base, and H for the slant height, then*

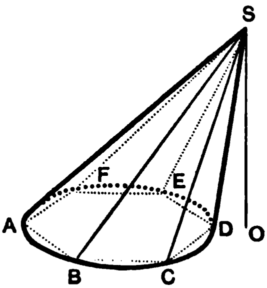
$$S = \pi r H.$$

1039. EXERCISE. The altitude of a cone of revolution is 12 in., and the radius of its base is 9 in. Find its lateral area.

1040. EXERCISE. The slant height of a cone of revolution is equal to the diameter of its base. The lateral area is 25.1328 sq. ft. Find the slant height.

PROPOSITION XIII. THEOREM

1041. *The volume of a circular cone is equal to one third the product of its base by its altitude.*



[The proof is left to the student.]

1042. Corollary. *If v stands for the volume, a for the altitude, and r for the radius of the base, then*

$$v = \frac{1}{3}\pi r^2 a.$$

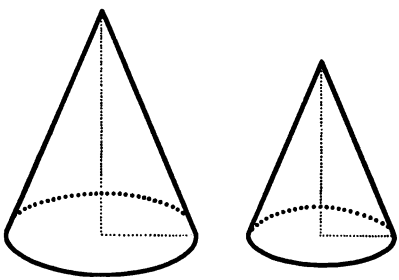
1043. Exercise. The volume of a circular cone is 100 cu. in. Its altitude is 25 ft. What is the area of its base?

1044. Exercise. The slant height of a cone of revolution is 25 ft., and the radius of its base is 20 ft. Find its volume.

1045. Definition. Similar cones of revolution are cones generated by similar right-angled triangles revolved about homologous legs as axes.

Proposition XIV. Theorem

1046. *The lateral or entire areas of two similar cones of revolution are to each other as the squares of their altitudes or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their altitudes or as the cubes of the radii of their bases.*



[Proof similar to that of § 1022.]

1047. Exercise. The volume of one of two similar cones of revolution is 125 times the volume of the other. Compare their surfaces.

1048. Exercise. A cone of revolution is cut into two portions by a plane parallel to the base. The portion with the vertex is $\frac{1}{7}$ of the remaining part. If the altitude is 8 in., how far from the vertex did the cutting plane pass?

1049. Definitions. A *truncated cone* is the portion of a cone included between its base and a plane cutting all of its elements.

The *frustum* of a cone is the portion of a cone included between its base and a plane parallel to its base.

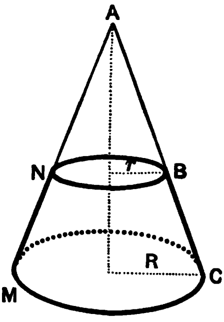
The base of the cone and the parallel section are called the *bases of the frustum.*

The *altitude of the frustum* is the perpendicular distance between the bases. The portion of an element included between the parallel bases of the frustum of a right circular cone is its *slant height.*

Proposition XV. Theorem

1050. *The lateral area of the frustum of a cone of revolution is equal to one half the sum of the circumferences of its bases multiplied by its slant height.*

Let r denote the radius of the upper base of the frustum, R the radius of the lower base, and BC the slant height.



To Prove the lateral area of the frustum $= \frac{1}{2}(2\pi R + 2\pi r)BC.$

Proof.

$$\frac{AB}{AC} = \frac{r}{R} = \frac{2\pi r}{2\pi R}.$$

$$AB \cdot 2\pi R = AC \cdot 2\pi r.$$

$$(AC - BC)\,2\pi R = (AB + BC)\,2\pi r.$$

$$\tfrac{1}{2}AC \cdot 2\pi R - \tfrac{1}{2}AB \cdot 2\pi r = \tfrac{1}{2}BC(2\pi R + 2\pi r).$$

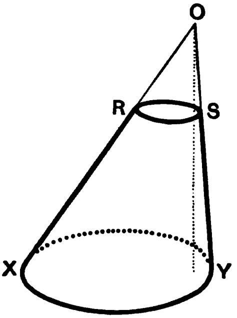
But $\frac{1}{2}AC \cdot 2\pi R$ is the lateral area of the cone $A\text{–}CM$, and $\frac{1}{2}AB \cdot 2\pi r$ is the lateral area of the cone $A\text{–}BN$, and their difference is the lateral area of the frustum.

$\therefore$ lateral area of frustum $= \frac{1}{2}(2\pi R + 2\pi r)BC.$ Q.E.D.

1051. Corollary. *The lateral area of the frustum of a cone of revolution is equal to the slant height multiplied by the circumference of a section midway between the bases.*

Proposition XVI. Theorem

1052. *The volume of the frustum of a circular cone is equal to the sum of its bases and a mean proportional between them multiplied by one third of the altitude of the frustum.*



Let XS be a frustum of the cone $O\text{-}XY$, and let B denote the area of the lower base, b the area of the upper base, and a the altitude of the frustum.

To Prove vol. $XS = \frac{1}{3}a(B + b + \sqrt{B \times b})$.

[See proof of § 972.]

1053. Corollary. *The volume of the frustum of a circular cone is equivalent to the sum of the volumes of three cones whose common altitude is the altitude of the frustum, and whose bases are the upper base, the lower base, and a mean proportional between them.*

1054. Exercise. The altitude of the frustum of a circular cone is 10 ft. The radii of upper and lower bases are 4 ft. and 9 ft. respectively. Find the volume of the frustum.

1055. Exercise. **If the cone in § 1054 is a cone of revolution, find the lateral area of the frustum.**

EXERCISES

1. The volume of a circular cone is a cu. in. and its altitude is b in. Find the radius of its base.

2. If the altitude of a cylinder of revolution is equal to the diameter of its base, its volume is equal to the product of its total surface by one third of the radius of the base.

3. The diameter of a well is 8 ft. Its depth is 10 ft. How many gallons of water will it hold?

4. The slant height of a right circular cone is equal to the diameter of its base. Compare the area of its base with its convex surface.

5. A cone is cut by two parallel planes. Show that the areas of the two sections are to each other as the squares of their distances from the vertex.

6. A cylindrical vessel contains a cu. in. Its height is b in. Find the diameter of its base.

7. Pass a plane parallel to the base of a cone cutting off a section whose area is equal to $\frac{1}{4}$ the base of the cone.

8. Divide a cone into halves by a plane parallel to the base of the cone.

9. The circumference of the base of a right cylinder is a in. The altitude of the cylinder is b in. Find the convex surface and the volume.

10. The radii of the bases of the frustum of a circular cone are 6 in. and 10 in. respectively. Its altitude is 8 in. Find its volume and its convex surface.

11. The intersection of two planes tangent to a cylinder is parallel to an element.

12. The number of cubic inches in the volume of a certain right cylinder is the same as the number of square inches in its convex surface. Find the radius of its base.

13. The volume of a cone is 400 cu. in. and its altitude is 48 in. Pass a plane, parallel to the base, cutting a section whose area is 9 sq. in.

14. The altitude of one of two similar cylinders of revolution is 5 times the altitude of the other. Compare their convex surfaces and their volumes.

15. The altitudes of two equivalent right cylinders are as 4 is to 7. If the diameter of the first is $3\frac{1}{2}$ ft., what is the diameter of the second?

16. The altitude of a cone of revolution is a ft. and the radius of its base is b ft. Find the dimensions of a similar cone 5 times as large.

17. The lateral area of a cylinder of revolution is equal to the area of a circle whose diameter is a mean proportional between the altitude and the diameter of the base of the cylinder.

18. The volumes of two similar cones of revolution are to each other as 125 : 216. How do their convex surfaces compare?

19. A right circular cone, whose slant height is equal to the diameter of its base, has the same base and altitude that a right cylinder has. Compare the convex surfaces of the cone and the cylinder.

20. The diameter of a right circular cylinder is 10 ft. and its altitude is 8 ft. What is the edge of an equivalent cube?

21. The altitude of a cone of revolution is three times the radius of its base. Its lateral area is 200 sq. in. Find its altitude and the radius of its base.

22. Pass a plane, parallel to the base of a cone, cutting off a cone whose volume is one third of the volume of the remaining frustum.

BOOK IX

1056. Definitions. A *sphere* is a solid bounded by a surface, all the points of which are equally distant from a point within called the *center*.

A *radius* of a sphere is a straight line drawn from the center to the surface.

A *diameter* of a sphere is a straight line drawn through the center and terminating in the surface.

A line or a plane is *tangent* to a sphere if it has one point and only one point in common with the surface of the sphere.

Two spheres are tangent to each other when their surfaces have one and only one point in common.

A polyhedron is *inscribed in a sphere* if all of its vertices are in the surface of the sphere. In this case the sphere is said to be *circumscribed about the polyhedron*.

A polyhedron is *circumscribed about a sphere* if all of its faces are tangent to the sphere. In this case, the sphere is said to be *inscribed in the polyhedron*.

It follows from the definition of a sphere that *all radii of the same sphere are equal*.

It can be shown that *spheres having equal radii are equal;* for they can be so placed that their surfaces will coincide. Conversely, *equal spheres have equal radii*.

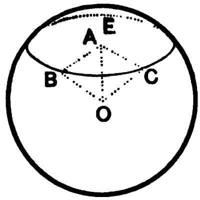
A sphere may be generated by revolving a semicircle about its diameter as an axis.

Proposition I. Theorem

1057. *Every section of a sphere made by a plane is a circle.*

Let BCE be a section made by a plane cutting the sphere whose center is O.

To Prove BCE a circle.



Proof. Draw $OA \perp$ to the plane BCE.

From B and C, any two points in the perimeter of BCE, draw BO, CO, BA, and CA.

$$BO = CO \ (?)$$

$$BA = CA \ (?)$$

All points of BCE are equally distant from A. (?)

$\therefore$ BCE is a circle. Q.E.D.

1058. DEFINITIONS. Any section made by a plane passing through the center of a sphere is a *great circle* of the sphere.

A *small circle* is a section made by a plane that does not pass through the center of the sphere.

A diameter of the sphere that is perpendicular to the plane of a circle of the sphere is the *axis* of that circle, and the extremities of the diameter are the *poles* of the circle.

1059. COROLLARY I. *The axis of a circle passes through the center of the circle.*

1060. COROLLARY II. *All great circles of a sphere are equal.*

1061. COROLLARY III. *Circles of a sphere made by planes equally distant from the center of the sphere are equal, and conversely.*

1062. COROLLARY IV. *Of two unequal circles, the smaller is at the greater distance from the center of the sphere, and conversely.*

1063. COROLLARY V. *Any two great circles of a sphere bisect each other.*

1064. COROLLARY VI. *Every great circle of a sphere bisects the sphere and its surface.*

1065. COROLLARY VII. *Through two given points on the surface of a sphere, not the extremities of a diameter, the arc of a great circle less than a semicircumference can be drawn, and* *only one.* [Pass a plane through the two points and the center of the sphere.]

1066. Corollary VIII. *Through any three points on the surface of a sphere one circumference can be drawn, and only one.*

1067. Exercise. Parallel circles of a sphere have the same axis and the same poles.

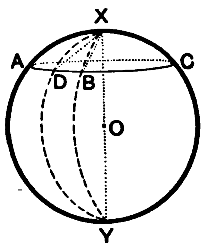
1068. Exercise. If the planes of two great circles are perpendicular to each other, each circle passes through the poles of the other.

1069. Exercise. A plane cuts a sphere at a distance of 3 in. from the center. If the radius of the sphere is 5 in., what is the diameter of the small circle?

1070. Definition. The distance between two points on the surface of a sphere is measured on the arc of a great circle joining them.

Proposition II. Theorem

1071. *All points on the circumference of a circle of a sphere are equally distant from each of its poles.*



Let ABC be any circle on the sphere and XY be its axis.

To Prove all points on the circumference ABC equally distant from X, and also equally distant from Y.

Proof. Let D and B be any two points in the circumference ABC. Pass arcs of great circles through X and D and through X and B. (§ 1065.)

Show that the chords XD and XB are equal,

whence arc $XD =$ arc XB.

Similarly, arc $YD =$ arc YB.

Since D and B are any two points on the circumference, all points on the circumference are equally distant from each of the poles. Q.E.D.

1072. Definition. The *polar distance* of a circle of a sphere is the distance of the nearer of its poles from its circumference.

1073. Corollary I. *All points on the circumference of a great circle of a sphere are at a quadrant's distance from either of its poles.*

1074. Corollary II. *If a point on the surface of a sphere is a quadrant's distance from each of two points, not extremities of a diameter, in the circumference of a great circle of the sphere, it is the pole of that great circle.*

1075. Exercise. If a point on the surface of a sphere is equally distant from three points on the circumference of a circle of the sphere, that point is the pole of the circle.

1076. Exercise. The distance of the plane of a small circle from the center of a sphere is one half the radius of the sphere. If the diameter of the sphere is 12 in., find the polar distance of the small circle in degrees and in inches.

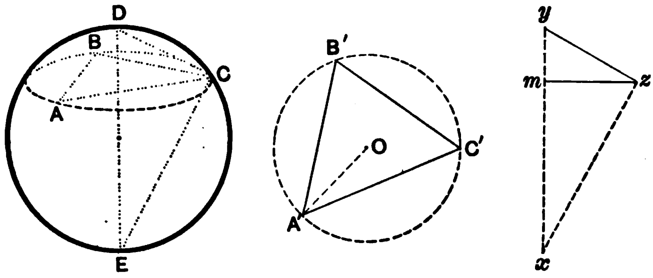
1077. Exercise. The polar distances of equal circles on the same sphere are equal.

1078. Scholium. Corollary II. suggests the plan for describing with compasses a great circle arc that shall pass through two given points on a material sphere. Using the points as centers and with opening in compasses equal to the chord of a quadrant, describe two intersecting arcs. This point of intersection is the pole of the great circle.

To determine the chord of a quadrant, the radius of the sphere must be known. Proposition III. finds this.

Proposition III. Problem

1079. *Given a material sphere, to find its diameter.*



Let DCE be the given sphere.

Required to find its diameter.

Using any point D on the surface of the sphere as a pole, describe any circumference ABC.

Take any three points on the circumference, as A, B, and C, and with compasses measure the chords AB, BC, and CA. Construct the $\triangle A'B'C'$, having for sides the lengths of the three chords.

Circumscribe a circle about $A'B'C'$.

Construct the R. A. $\triangle myz$, having the hypotenuse yz equal to the chord DC and mz equal to the radius $A'O$.

Complete the R. A. $\triangle yzx$, right-angled at z.

Prove $\triangle xyz$ equal to $\triangle DCE$, in which DE is a diameter of the sphere; whence $xy = DE$. Q.E.F.

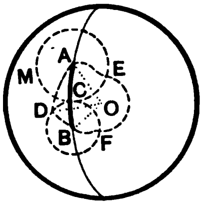
1080. Exercise. Construct a circle equal to a great circle of a material sphere.

1081. Construct a circle equal to any given small circle on the surface of a material sphere.

Proposition IV. Theorem

1082. *The shortest line joining two points on the surface of a sphere is the arc of a great circle, less than a semicircumference, that joins them.*

Let A and B be any two points on the surface of a sphere whose center is O, and let AB be the arc of a great circle, less than a semicircumference, joining them.



To Prove AB is the shortest line on the surface joining A and B.

Proof. Let C be any point in AB.

With A and B as poles and with AC and BC as polar distances, describe the Ⓢ CEM and DCF.

C is the only point common to the two circles. For through any other point of DCF, as D, draw the great circle arcs DA and DB. Draw the radii of the sphere AO, BO, and DO.

$$\angle BOD + \angle DOA > \angle AOB. \quad (?)$$

$$DB + DA > AC + CB. \quad (?)$$

$$DA > AC. \quad (?)$$

By § 1071 D cannot be on the circumference CEM, and the two circles have only C in common.

The shortest line joining A and B must pass through C. For join A and B by any line not passing through C, as $AEFB$.

No matter what the character of AE is, a similar line can be drawn from A to C, and a line similar to BF can be drawn from B to C. Hence a line can be drawn from A to B and passing through C that will exactly equal $AE + BF$. This line will be less than $AEFB$.

∴ the shortest line from A to B must pass through C.

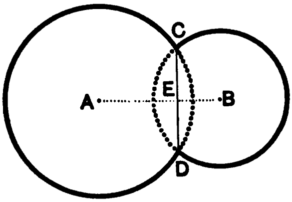
But C is *any* point on AB. Therefore the shortest line from A to B must pass through *every* point of AB; and AB is the shortest line joining A and B. Q.E.D.

1083. Scholium. The use of the term *distance* in spherical geometry is analogous to that of plane geometry, for here, too, *distance* means *shortest distance*, on the surface.

1084. Exercise. The longest arc of a circle of a sphere, joining two points on the surface, is the arc of a great circle, greater than a semicircumference, that joins them.

Proposition V. Theorem

1085. *The intersection of the surfaces of two spheres is the circumference of a circle whose center is on the line joining the centers of the spheres, and whose plane is perpendicular to that line.*



Let the spheres whose centers are A and B intersect.

To Prove that the intersection of their surfaces is the circumference of a circle whose center is on AB and whose plane is perpendicular to AB.

Proof. Pass any plane through AB. The intersection of this plane with the surface of the spheres is two circumferences intersecting at C and D. AB bisects the chord CD at right angles. (?)

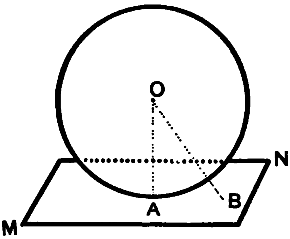
If the plane of the circumferences be revolved about AB as an axis, the circles will generate the spheres and the point C will describe the line of intersection of their surfaces. But C describes the circumference of a circle whose center is E and whose plane is perpendicular to AB. (?) Q.E.D.

1086. Exercise. What is the locus of the centers of spheres having a given radius and tangent to a given sphere?

1087. Exercise. What is the locus of the centers of spheres having a given radius and tangent to two given spheres?

Proposition VI. Theorem

1088. *A plane that is perpendicular to the radius of a sphere at its outer extremity is tangent to the sphere; and conversely a plane that is tangent to a sphere is perpendicular to a radius drawn to the point of tangency.*



[The proof is left to the student. See § 307.]

1089. Exercise. If a plane is tangent to a sphere, a line drawn perpendicular to the plane at the point of tangency passes through the center of the sphere.

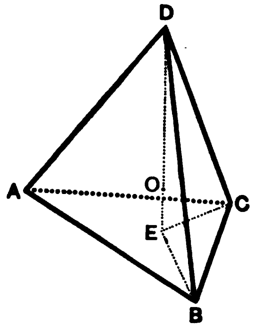
1090. Exercise. If two spheres are tangent either externally or internally, their centers and the point of tangency are in the same line. [See § 326.]

1091. Exercise. State the proposition for spheres analogous to Prop. XVII, Bk. II.

1092. Exercise. If two straight lines are tangent to a sphere at the same point, their plane is tangent to the sphere.

Proposition VII. Theorem

1093. *A sphere can be inscribed in any tetrahedron.*



Let *ABCD* be any tetrahedron.

To Prove that a sphere can be inscribed in it.

Proof. Let *DE* be the intersection of the planes that bisect the dihedral angles whose edges are *DC* and *DB*. Every point on *DE* is equally distant from the faces *ADB*, *ADC*, and *DBC*. (?)

Let the plane bisecting the dihedral angle whose edge is *BC*, intersect *DE* at *O*.

Show that *O* is equally distant from all the faces of the tetrahedron, and is the center of the inscribed sphere. Q.E.D.

1094. Exercise. The planes that bisect three dihedral angles of a tetrahedron whose edges meet at a common point, meet in a common line.

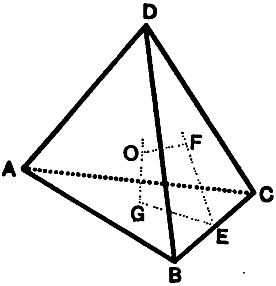
1095. Exercise. The planes that bisect the six dihedral angles of a tetrahedron all pass through a common point.

1096. Exercise. What is the locus of the centers of spheres that are tangent to the faces of a trihedral angle?

1097. Exercise. Describe a sphere tangent to four intersecting planes.

PROPOSITION VIII. THEOREM

1098. ***A sphere can be circumscribed about any tetrahedron.***



Let *ABCD* be any tetrahedron.

To Prove that a sphere can be circumscribed about it.

Proof. Let *GE*, in the plane *ABC*, be ⊥ to *BC* and bisect it, and let *G* be the center of the circle that can be circumscribed about *ABC*.

Let *F* be the center of the circle circumscribed about *DBC*.

The plane of *GE* and *EF* is ⊥ to both *ABC* and *DBC*. (?)

At *G* and *F* erect ⊥s to the planes *ABC* and *DBC* respectively.

Show that these ⊥s lie in the plane *GEF* and will meet at some point *O*.

Show that *O* is equally distant from *A*, *B*, *C*, and *D*. Q.E.D.

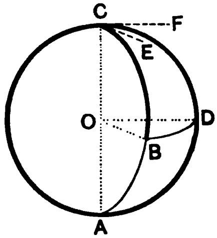
1099. EXERCISE. What is the locus of the centers of spheres whose surfaces pass through three given points?

1100. EXERCISE. Describe a spherical surface passing through four points in space.

1101. DEFINITION. The angle formed by two intersecting curves is the angle of the tangents to the curves at their point of intersection. The angle formed by the intersection of two great circle arcs is a *spherical angle*.

Proposition IX. Theorem

1102. *The measure of a spherical angle is the arc of a great circle described with the vertex of the spherical angle as a pole and included between its sides.*



Let the great circle arcs ABC and ADC intersect at C, and let BD be the arc of a great circle described with C as a pole.

To Prove that BD is the measure of angle DCB.

Proof. Draw the radii OB and OD and the tangents CE and CF.

Arcs CB and CD are quadrants. (?)

OB and OD are each $\perp$ to CA. (?)

$$\angle BOD = \angle ECF. \quad (?)$$

BD is the measure of angle C. (?) Q.E.D.

1103. Corollary I. *The angle formed by the intersection of two great circle arcs is equal to the dihedral angle formed by the planes of those arcs.*

1104. Corollary II. *Any great circle arc through the pole of a great circle is perpendicular to the great circle.*

1105. Corollary III. *Any great circle arc that is perpendicular to the arc of another great circle passes through its pole.*

1106. DEFINITIONS. A *spherical polygon* is a portion of the surface of a sphere, bounded by three or more arcs of great circles.

The arcs are the *sides* of the polygon; their angles are the *angles* of the polygon; and their points of intersection are the *vertices* of the polygon. The arc of a great circle joining any two non-adjacent vertices is a *diagonal* of the polygon.

The planes of the sides of a spherical polygon form a polyhedral angle at the center of the sphere. As the sides of the spherical polygon are the measures of the face angles of the polyhedral angle at the center, properties of these curves may be deduced from known relations of the face angles.

A spherical polygon, corresponding to a convex polyhedral angle at the center of the sphere, is a *convex* polygon.

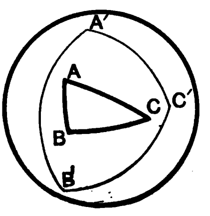
All spherical polygons used in this work may be regarded as convex polygons, unless the contrary is specially stated.

A spherical polygon having three sides is called a *spherical triangle.*

Spherical triangles are isosceles, equilateral, right-angled, etc., under the same conditions that plane triangles are isosceles, equilateral, right-angled, etc.

Two polygons are *equal* if their parts (sides and angles) are equal each to each, and arranged in the same order. In this case one polygon can be placed to coincide with the other.

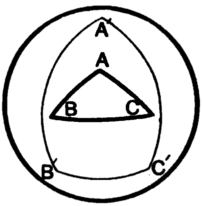
Two polygons are *symmetrical* if their parts are equal each to each, but in reverse order. Symmetrical polygons cannot in general be made to coincide.



If from the vertices of a spherical triangle ABC as poles, circumferences of great circles be described, these circumferences will divide the surface of the sphere into eight triangles. One of these triangles is the *polar triangle* of the triangle ABC. This triangle ($A'B'C'$) may be determined in the following manner. The vertex A', formed by the intersection of curves whose poles are B and C, is on the same side of BC that vertex A is, and less than a quadrant's distance from A. Similarly C' and C are on the same side of AB and less than a quadrant's distance from each other; and B' and B are on the same side of AC and less than a quadrant's distance from each other.

Proposition X. Theorem

1107. *If one spherical triangle is the polar triangle of another, the second is the polar triangle of the first.*



Let $A'B'C'$ be the polar triangle of ABC.

To Prove ABC the polar triangle of $A'B'C'$.

Proof. Since C is the pole of $A'B'$, B' is a quadrant's distance from C. Since A is the pole of $B'C'$, B' is a quadrant's distance from A.

Since B' is a quadrant's distance from both A and C, B' is the pole of AC.

Similarly C' is the pole of AB, and A' is the pole of BC.

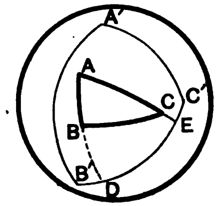
$\therefore$ ABC is the polar triangle of $A'B'C'$. Q.E.D.

1108. Exercise. What triangle is its own polar triangle?

1109. Exercise. Will the sides of a triangle ever intersect the sides of its polar triangle?

PROPOSITION XI. THEOREM

1110. *Each angle of a spherical triangle is measured by the supplement of the side of which it is the pole, in the polar triangle.*



Let ABC be any spherical triangle and $A'B'C'$ its polar triangle.

To Prove that angle A is measured by the supplement of $B'C'$.

Proof. Prolong AB and AC until they meet $B'C'$ at D and E respectively.

DE is the measure of angle A. (?)

$B'E$ and DC' are each quadrants.

$DE = 2$ quadrants $- B'C'$.

$\therefore \angle A \sim$ supplement of $B'C'$.

Similarly show that $\angle A' \sim$ supplement of BC. Q.E.D.

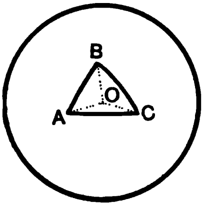
1111. Exercise. The sides of a spherical triangle are 79°, 127°, and 84°. How many degrees in each angle of its polar triangle?

1112. Exercise. The angles of a spherical triangle are 85°, 74°, and 126°. How many degrees in each side of its polar triangle?

PROPOSITION XII. THEOREM

1113. *One side of a spherical triangle is less than the sum of the other two sides.*

Let ABC be any spherical triangle.



To Prove $AB + BC > AC$.

Proof. Draw the radii of the sphere AO, BO, CO.

$\angle AOB + \angle BOC > \angle AOC$. (?)

$\therefore AB + BC > AC$. (?) Q.E.D.

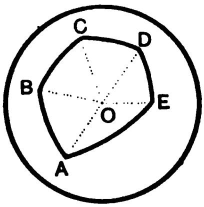
1114. Exercise. One side of a spherical polygon is less than the sum of the other sides.

1115. Exercise. The sum of the arcs of great circles drawn from any point within a triangle to the extremities of a side, is less than that of the remaining sides of the triangle.

1116. Exercise. If from a point within a triangle arcs of great circles are drawn to the three vertices, their sum is less than the perimeter of the triangle and greater than the semiperimeter.

Proposition XIII. Theorem

1117. *The sum of the sides of any spherical polygon is less than the circumference of a great circle.*



Let $ABCDE$ be any spherical polygon.

To Prove $AB + BC + CD + DE + EA <$ the circumference of a great circle.

Proof. Draw the radii of the sphere AO, BO, CO, DO, and EO.

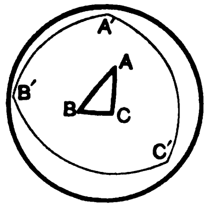
$\angle AOB + \angle BOC + \angle COD + \angle DOE + \angle EOA < 4$ R.A.'s. (?)

$\therefore$ $AB + BC + CD + DE + EA <$ the circumference of a great circle. Q.E.D.

1118. Exercise. Between what limits does the perimeter of a spherical polygon lie? [It is less than the circumference of a great circle and greater than what?]

PROPOSITION XIV. THEOREM

1119. ***The sum of the angles of a spherical triangle is more than two right angles and less than six right angles.***



Let ABC be any spherical triangle.

To Prove $\angle A + \angle B + \angle C > 2$ R.A.'s,

and $\angle A + \angle B + \angle C < 6$ R.A.'s.

Proof. Let $A'B'C'$ be the polar triangle of ABC, and designate its sides by a', b', and c'.

$$\angle A = 180° - a', \quad \angle B = 180° - b', \quad \angle C = 180° - c'. \quad (?)$$

$$\angle A + \angle B + \angle C = 540° - (a' + b' + c'). \quad (?) \qquad (1)$$

$$a' + b' + c' < 360°. \quad (?) \qquad (2)$$

From (1) deduce

$$\angle A + \angle B + \angle C < 540° \text{ or } 6 \text{ R.A.'s}.$$

From (1) and (2) deduce

$$\angle A + \angle B + \angle C > 180° \text{ or } 2 \text{ R.A.'s}. \qquad \text{Q.E.D.}$$

1120. COROLLARY. *A spherical triangle may contain two right angles, or even three right angles. It may contain three obtuse angles.*

1121. DEFINITIONS. A triangle containing two right angles is a *birectangular triangle.* A triangle containing three right angles is a *trirectangular triangle.*

1122. EXERCISE. In a birectangular triangle the sides opposite the right angles are quadrants.

1123. EXERCISE. If two sides of a triangle are quadrants, the angles opposite them are right angles.

1124. EXERCISE. The sides of a trirectangular triangle are quadrants.

1125. EXERCISE. If three sides of a triangle are quadrants, the three angles are right angles.

PROPOSITION XV. THEOREM

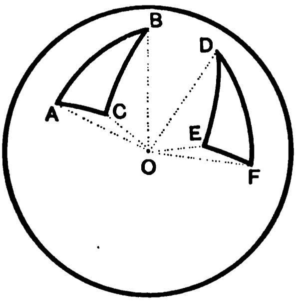
1126. *On the same or equal spheres, triangles that have*

1. *Two sides and the included angle of one equal respectively to two sides and the included angle of the other; or*

2. *Two angles and the included side of one equal respectively to two angles and the included side of the other; or*

3. *Three sides of one equal respectively to the three sides of the other;*

Have their remaining parts equal, and the triangles are either equal or symmetrical.



Proof. Draw radii of the sphere to the vertices of the triangles, forming two trihedral angles.

By application of §§ 899, 900, and 901, show that the remaining parts of the triangles are equal.

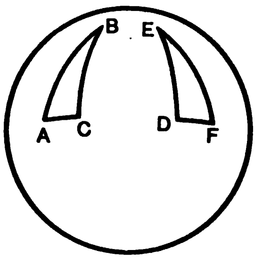
If the equal parts are arranged in the same order, the triangles can be made to coincide, and are consequently equal.

If the equal parts are arranged in reverse order, the triangles are symmetrical. In this case they cannot be made to coincide unless the triangles are isosceles.

Symmetrical isosceles triangles are equal.

PROPOSITION XVI. THEOREM

1127. *On the same or equal spheres, spherical triangles that are mutually equiangular are mutually equilateral, and are either equal or symmetrical.*



Let *ABC* and *DEF* be two mutually equiangular triangles.

To Prove *ABC* and *DEF* mutually equilateral, and either equal or symmetrical.

Proof. If *ABC* and *DEF* are mutually equiangular, their polar triangles are mutually equilateral. (?)

If their polar triangles are mutually equilateral, they are also mutually equiangular. (?)

If their polar triangles are mutually equiangular, *ABC* and *DEF* are mutually equilateral. (?)

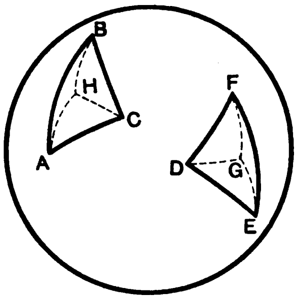
If the equal parts of the two triangles *ABC* and *DEF* are arranged in the same order, the triangles are equal; but if the equal parts are arranged in reverse order, the triangles are symmetrical. Q.E.D.

1128. EXERCISE. Spherical triangles corresponding to vertical trihedral angles at the center of a sphere are symmetrical.

1129. EXERCISE. If two trihedral angles have the dihedral angles of one equal respectively to the dihedral angles of the other, their plane angles are equal each to each.

Proposition XVII. Theorem

1130. *Symmetrical spherical triangles are equivalent.*



Let ABC and DEF be two symmetrical triangles having $AB = FE$, $AC = DE$, and $BC = FD$.

To Prove $\triangle$s ABC and DEF equivalent.

Proof. Let H be the pole of a small circle passing through A, B, and C, and G the pole of a small circle passing through D, E, and F. Draw the great circle arcs HA, HB, HC, GD, GE, and GF.

The chords of AB, AC, and BC are equal respectively to the chords of FE, DE, and DF. (?)

The plane $\triangle ABC =$ the plane $\triangle DEF$. (?)

Small circle $ABC =$ small circle DEF. (?)

The polar distances HA, HB, and HC are equal respectively to the polar distances GE, GF, and GD. (?)

The isosceles $\triangle HAB =$ the isosceles $\triangle GFE$. (?)

Similarly, $\triangle HAC = \triangle GDE$ and $\triangle HBC = \triangle GFD$.

$\therefore$ $\triangle$s ABC and DEF are equivalent. Q.E.D.

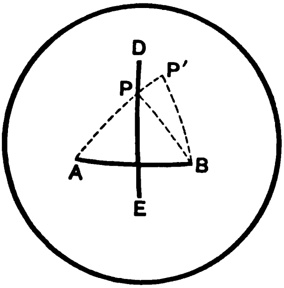
1131. Exercise. Two symmetrical spherical polygons are equivalent, for they can be divided into symmetrical triangles.

Proposition XVIII. Theorem

1132. *If the arc of a great circle is perpendicular to a second arc at its middle point,*

I. *Any point on the perpendicular is equally distant from the extremities of the second arc;*

II. *Any point without the perpendicular is unequally distant from the extremities of the second arc.*



[Proof similar to that of the corresponding plane proposition.]

1133. Corollary I. *The perpendicular contains all points that are each equally distant from the extremities of the second arc.*

1134. Corollary II. *If two points on an arc, not the extremities of a diameter, are each equally distant from the extremities of a second arc, the first arc is perpendicular to the second and bisects it.*

1135. Corollary III. *Through a given point to draw a perpendicular to a given arc.* [Two cases.]

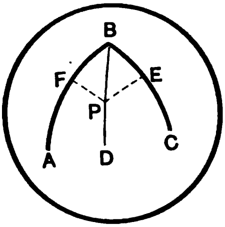
1136. Corollary IV. *To bisect a given arc.*

1137. Exercise. From a given point (not the pole) only one perpendicular can be drawn to a given arc. [Two cases.]

1138. Exercise. The shortest distance from a point to an arc is the perpendicular distance.

PROPOSITION XIX. THEOREM

1139. *Any point on the bisector of a spherical angle is equally distant from the sides of the angle.*



Let BD be the bisector of the spherical angle ABC.

To Prove that any point in BD as P is equally distant from AB and BC.

Proof. Draw $PF \perp$ to AB.

Lay off $BE = BF$ and draw PE.

$\triangle$ PBF and PBE have two sides and the included angle of one equal respectively to two sides and the included angle of the other, whence $\angle F = \angle E$ and $PF = PE$.

PE is therefore perpendicular to BC, and P is equally distant from BA and BC. Q.E.D.

1140. COROLLARY I. *Any point without the bisector of an angle is unequally distant from the sides of the angle.*

[Proof similar to that of the corresponding plane proposition.]

1141. COROLLARY II. *The bisector of an angle is the locus of points that are each equally distant from the sides of the angle.*

1142. COROLLARY III. *Bisect a given spherical angle.*

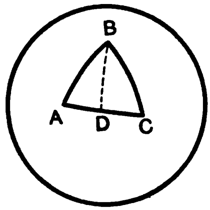
1143. COROLLARY IV. *Construct an angle equal to a given spherical angle.*

1144. EXERCISE. Construct an angle equal to double a given angle.

1145. EXERCISE. Construct an angle equal to one half a given angle.

Proposition XX. Theorem

1146. *If two sides of a spherical triangle are equal, the angles opposite them are equal.*



Let ABC be a spherical triangle having AB equal to BC.

To Prove $\angle A = \angle C$.

[Draw BD to the middle point of AC, and apply § 1126.]

Converse. If two angles of a triangle are equal, the sides opposite them are equal.

Let ABC have $\angle A = \angle C$.

To Prove $AB = BC$.

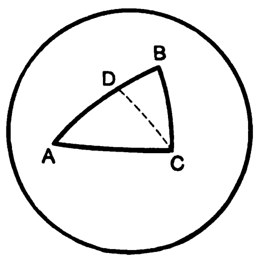
Proof. The polar triangle of ABC has two sides equal. (?)
The polar triangle of ABC has two angles equal. (?)
ABC has its sides AB and BC equal. (?) Q.E.D.

1147. Corollary. *An equilateral triangle is equiangular, and conversely, an equiangular triangle is equilateral.*

1148. Exercise. If the $\triangle ABC$ has $\angle A = \angle C$, and FA and FC are bisectors of angles A and C respectively, prove $FA = FC$.

Proposition XXI. Theorem

1149. *If two angles of a spherical triangle are unequal, the sides opposite them are unequal, the greater side lying opposite the greater angle.*



Let ABC be a triangle having $\angle C$ greater than $\angle A$.

To Prove $AB > BC$.

Draw DC, making $\angle DCA = \angle A$, and prove as in corresponding plane proposition.

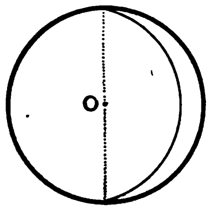
Converse. If two sides of a spherical triangle are unequal, the angles opposite them are unequal, the greater angle lying opposite the greater side.

[Prove indirectly.]

1150. Exercise. Prove the converse to this proposition by using the polar triangle.

1151. Exercise. If the $\triangle ABC$ has $\angle C > \angle A$, and FA and FC are bisectors of $\angle A$ and $\angle C$ respectively, prove $FA > FC$.

1152. Definitions. A *lune* is a portion of the surface of a sphere included between semicircumferences of great circles.



The *angle of a lune* is the angle made by its bounding arcs.

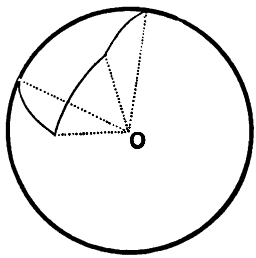
As the angle of a lune varies from 0° to 360°, the area of the lune varies from zero to the surface of the sphere.

A *spherical wedge* is a portion of the sphere bounded by a lune and the planes of its arcs. The lune is called the *base* of the wedge.

Lunes with equal angles are equal, for they can be made to coincide.

Wedges with equal bases are equal, for they can be made to coincide.

A *spherical pyramid* is a portion of the sphere bounded by a spherical polygon and the planes of its sides. The spherical polygon is called the *base* of the spherical pyramid.



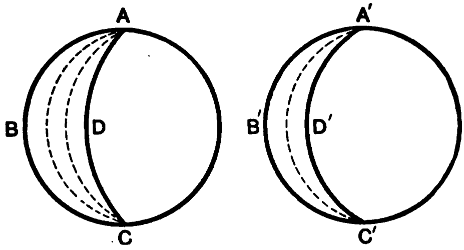
Spherical pyramids with equal bases are equal, for they can be made to coincide.

Spherical pyramids with symmetrical bases are equivalent. [Proof similar to that of § 1130.]

Proposition XXII. Theorem

1153. *Two lunes on the same or equal spheres are to each other as their angles.*

Case I. When their angles are commensurable.



Let $ABCD$ and $A'B'C'D'$ be two lunes on equal spheres and with commensurable angles BAD and $B'A'D'$.

To Prove $$\frac{ABCD}{A'B'C'D'} = \frac{\angle BAD}{\angle B'A'D'}.$$

Proof. Let the common unit of measure be contained in $\angle BAD$ m times and in $\angle B'A'D'$ n times.

Whence $$\frac{\angle BAD}{\angle B'A'D'} = \frac{m}{n}. \quad (1)$$

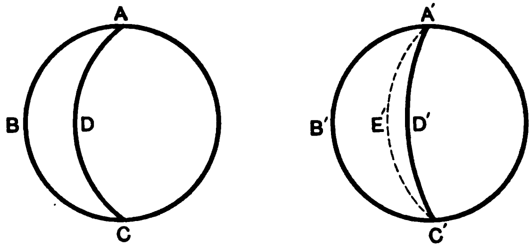
Divide $\angle BAD$ into m equal parts, each equal to the unit of measure, and $\angle B'A'D'$ into n equal parts, each equal to the unit of measure. Prolong the arcs of division to C and C' respectively.

The lune $ABCD$ is divided into m lunes and $A'B'C'D'$ into n lunes, and these lunes are equal to each other. (?)

$$\therefore \frac{ABCD}{A'B'C'D'} = \frac{m}{n}, \quad (2)$$

and $$\frac{ABCD}{A'B'C'D'} = \frac{\angle BAD}{\angle B'A'D'}. \quad (?)$$ Q.E.D.

CASE II. When their angles are incommensurable.



Let $\angle s\, BAD$ and $B'A'D'$ be incommensurable.

To Prove $$\frac{ABCD}{A'B'C'D'} = \frac{\angle BAD}{\angle B'A'D'}.$$

Proof. Divide $ABCD$ into a number of equal parts and apply one of these parts to $A'B'C'D'$ as a unit of measure, and proceed as in § 344.

1154. COROLLARY I. *The area of a lune is to the surface of the sphere as the angle of the lune is to four right angles.*

1155. SCHOLIUM. Two great circles at right angles to each other divide the surface of the sphere into four equal lunes. If, using either point of intersection as a pole a third great circle be described, it will divide each lune into two trirectangular triangles.

The area of a trirectangular triangle is equal to one eighth of the surface of the sphere.

1156. COROLLARY II. *The area of a lune is to the area of a trirectangular triangle as twice the angle of the lune (expressed in right angles) is to one right angle.* *Whence*

The area of a lune is equal to the area of a trirectangular triangle multiplied by twice the number of right angles in the angle of the lune.

1157. Scholium. By a course of reasoning similar to that employed in proving Prop. XXII. and corollaries, the following principles relating to spherical wedges can be established:

1. The volumes of two spherical wedges, on the same or equal spheres, are to each other as their angles.
2. The volume of a wedge is to the volume of the sphere as the angle of the wedge is to four right angles.
3. The volume of a wedge is to the volume of a trirectangular pyramid [which is one eighth of the volume of the sphere (?)], as twice the angle of the wedge is to one right angle.
4. The volume of a wedge is equal to the volume of a trirectangular pyramid multiplied by twice the number of right angles in the angle of the wedge.

1158. Exercise. The volume of a sphere contains 200 cu. in. Find the volume of a wedge whose angle is 75°.

1159. Exercise. The volume of a wedge is 80 cu. ft. The volume of the sphere is 240 cu. ft. Find the angle of the wedge.

1160. Definition. The *spherical excess of a triangle* is the excess of the sum of its angles over two right angles. Thus, if A, B, and C represent the values of the angles of a triangle, then $(A + B + C - 2)$ R.A.'s is its spherical excess; or, if the angles of a triangle are 75°, 60°, and 103° respectively, then $\frac{75 + 60 + 103 - 180}{90}$ or $\frac{58}{90}$ R.A. is its spherical excess.

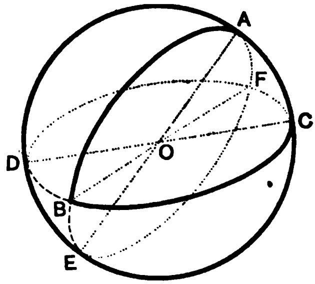
1161. Exercise. The angle of a lune is 45°. Show that its area is equal to that of a trirectangular triangle.

1162. Exercise. In the spherical triangle ABC, $\angle A = \frac{2}{5}$ R. A., $\angle B = 127°$, and $\angle C = 63°$. Find its spherical excess.

1163. Exercise. Find the area of a lune on a sphere whose surface is 160 sq. in., the angle of the lune being 75°.

Proposition XXIII. Theorem

1164. *The area of a spherical triangle is equal to the product of the number of right angles in its spherical excess by the area of a trirectangular triangle.*



Let ABC be any spherical triangle and let T denote the area of a trirectangular triangle.

To Prove

$$ABC = (A + B + C - 2)T.$$

Proof. Produce the sides of the triangle ABC until they become great circles. Draw the diameters BF, DC, and AE.

The area of the lune $CADB = 2\,C \times T$. (?) (1)

The area of the lune $ABEC = 2\,A \times T$. (?) (2)

The $\triangle$ ACF and DBE are equivalent. [Show that the trihedral angles $O\text{–}ACF$ and $O\text{–}DBE$ are symmetrical.]

$$\triangle ABC + \triangle DBE = \text{lune } ABCF = 2\,B \times T. \quad (?) \qquad (3)$$

Adding (1), (2), and (3),

Lune $CADB$ + lune $ABEC + \triangle ABC + \triangle DBE$

$$= (A + B + C)2\,T. \qquad (4)$$

The first member of (4) is equivalent to a hemisphere, or $4\,T$, increased by $2\triangle ABC$.

$$\therefore\ 4\,T + 2\triangle ABC = (A + B + C)2\,T.$$

Whence $$\triangle ABC = (A + B + C - 2)T.$$ Q.E.D.

1165. Corollary I. *The volume of a triangular spherical pyramid is equal to the product of the number of right angles in the spherical excess of its base by the volume of a trirectangular pyramid.*

[Proof similar to that of § 1164.]

1166. Corollary II. *The volumes of two triangular spherical pyramids of the same or equal spheres, are to each other as their bases.* [Use § 1165.]

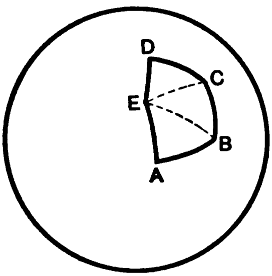
1167. Exercise. The angles of a spherical triangle are 80°, 130°, and 90°. The area of a trirectangular triangle is 40 sq. in. What is the area of the triangle?

1168. Exercise. The sum of the angles of a spherical polygon of n sides is greater than $(2n-4)$ R. A.'s and less than $2n$ R. A.'s.

1169. Definition. The spherical excess of a spherical polygon of n sides is the excess of the sum of its angles over $(2n-4)$ R.A.'s.

Proposition XXIV. Theorem

1170. *The area of a spherical polygon is equal to the product of the number of right angles in its spherical excess by the area of a trirectangular triangle.*



Let $ABCDE$ be a polygon of n sides and S denote the sum of its angles.

To Prove

area $ABCDE = [S-(2n-4)]T$.

Proof. From vertex E draw all the diagonals possible. They will divide the polygon into $(n-2)$ triangles.

The area of each triangle formed is equal to the product of the number of right angles in its spherical excess by the area of a trirectangular triangle.

The sum of the areas of the triangles = the area of the polygon, and the sum of the spherical excesses of the triangles = the spherical excess of the polygon.

$\therefore$ area $ABCDE = [S-(2n-4)]T$. (?) Q.E.D.

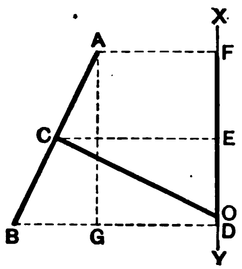
1171. Corollary I. *The volume of a spherical pyramid is equal to the product of the number of right angles in the spherical excess of its base by the volume of a trirectangular pyramid.*

1172. Corollary II. *The volumes of spherical pyramids of the same or equal spheres are to each other as their bases.*

1173. Exercise. The angles of a spherical quadrilateral are 103°, 157°, 90°, and 130°. The surface of the sphere contains 250 sq. in. Find the area of the quadrilateral.

Proposition XXV. Theorem

1174. *The surface generated by a straight line revolving about an axis in the same plane (but not crossing it) is equivalent to the product of its projection on the axis by the circumference whose radius is a perpendicular erected to the line at its middle point, and terminated by the axis.*



Let AB be the given line, XY the axis, FD the projection of AB on XY, and CO the perpendicular to AB at its middle point C and terminating in XY.

To Prove

surface $AB = FD \times 2\pi CO$.

Proof.

Surface $AB = 2\pi CE \times AB$. (?) (1)

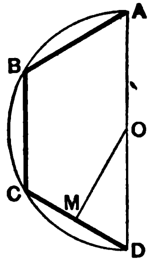
Show that △ BAG and CEO are similar.

Whence $CE \times AB = AG \times CO = FD \times CO$.

Substituting in (1), surface $AB = 2\pi CO \times FD$. Q.E.D.

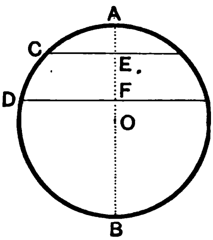
1175. Exercise. Show that this proposition is true if AB is parallel to XY, or if AB meets XY.

1176. Exercise. Show that this proposition is not true if AB crosses XY.



1177. Exercise. Show that the surface generated by revolving $ABCD$ (called a regular semiperimeter) about AD as an axis is equal to $AD \times 2\pi OM$. [OM is the apothem of the regular polygon.]

1178. Definitions. A *zone* is a portion of the surface of a sphere included between two parallel planes.



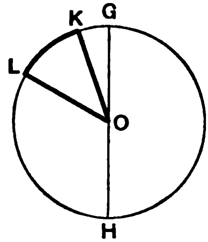
The distance between the parallel planes is the *altitude of the zone*, and the circumferences of the circles that bound it are its *bases*.

The surface of a sphere included between two parallel planes, if one of them is tangent to the sphere, is a *zone of one base*.

A *spherical segment* is a portion of a sphere included between two parallel planes. The distance between the planes is the *altitude of the segment*, and the sections of the sphere formed by the planes are its *bases*.

A portion of a sphere included between two parallel planes, one of which is tangent to the sphere, is a *spherical segment of one base*.

If the semicircle ADB be revolved about its diameter AB as an axis, the arc DC will generate a zone, and $DFEC$ will generate a spherical segment; DA will generate a zone of one base, and DFA will generate a spherical segment of one base.

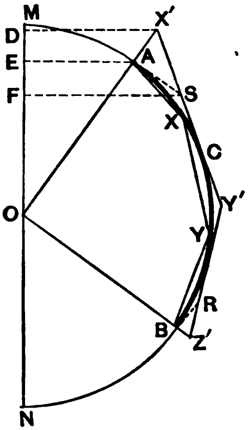


If the semicircle GLH be revolved about GH as an axis, the volume generated by a sector LOK is a *spherical sector.*

The surface generated by the arc LK is the *base* of the spherical sector.

PROPOSITION XXVI. THEOREM

1179. *If any portion of a semicircumference with a circumscribed broken line and an inscribed broken line be revolved about a diameter as an axis, the surface generated by the arc will be less than the surface generated by the circumscribed broken line, but greater than that generated by the inscribed broken line.*



Let the arc ACB with the broken lines $X'Y'Z'$ and $AXYB$ be revolved about MN as an axis.

I. **To Prove** surface $ACB <$ surface $X'Y'Z'$.

Proof. If $X'Y'Z'$ does not pass through the extremities of the arc AB, draw $AS \perp$ to the radius OA.

Draw SF, AE, and $X'D \perp$ to the diameter MN.

$$\text{Surface } AS = AS(SF + AE)\pi. \quad (?)$$

$$\text{Surface } X'S = X'S(SF + X'D)\pi. \quad (?)$$

Since $X'S > AS$ and $X'D > AE$ (?), surface $X'S >$ surface AS.

Similarly, surface $RZ' >$ surface RB.

$$\therefore \text{ surface } ASY'RB < \text{surface } X'Y'Z'. \quad (?)$$

That is, if the given circumscribed broken line does not pass through the extremities of the arc, a new circumscribed broken line which does pass through the extremities of the arc can always be found such that the surface generated by it is less than that generated by the given circumscribed line.

Now of all the surfaces ACB, $ASY'RB$, etc., that envelop the zone generated by ACB, there must be one whose area is a minimum.

Show that surface ACB is the minimum, and that

$$\text{surface } ACB < \text{surface } X'Y'Z'. \qquad \text{Q.E.D.}$$

II. **To Prove** surface $ACB >$ surface $AXYB$.

[Show that of all the surfaces $AXYB$, ACB, etc., enveloping the surface generated by $AXYB$, the surface $AXYB$ is the minimum.]

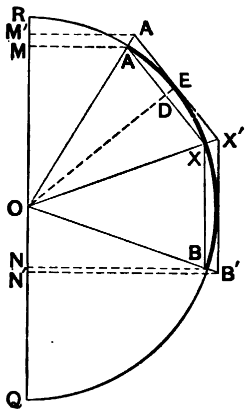
1180. Corollary. *If a semicircle with a regular inscribed semipolygon and a regular circumscribed semipolygon be revolved about its diameter as an axis, the surface of the sphere generated by the semicircumference will be greater than the surface generated by the inscribed semiperimeter, but less than that generated by the circumscribed semiperimeter.*

1181. Definitions. If an arc is divided into equal parts, the chords connecting the successive points of division form a *regular broken line* inscribed in the arc. Tangents parallel to these chords form a *regular circumscribed broken line.*

A regular broken line is not necessarily a part of the perimeter of a regular inscribed or circumscribed polygon.

PROPOSITION XXVII. THEOREM

1182. *If a portion of a semicircumference with a regular inscribed or circumscribed broken line be revolved about a diameter as its axis, and if the number of divisions of the broken line be indefinitely increased, the surface of the zone generated by the arc is the limit of the surface generated by the regular broken line.*



Let the arc AEB, with the broken lines AXB and $A'X'B'$, be revolved about the diameter RQ as an axis, and let the number of divisions be indefinitely increased.

To Prove that surface AEB is the limit of surface AXB and of surface $A'X'B'$.

Proof. Designate the surfaces $A'X'B'$, AEB, and AXB by S, S', and s respectively. Draw $A'M'$, AM, BN, and $B'N'$ $\perp$ to RQ.

$$\frac{S}{s} = \frac{M'N' \times 2\pi OE}{MN \times 2\pi OD} = \frac{M'N' \times OE}{MN \times OD}. \quad (?)$$

The polygons $M'A'X'B'N'$ and $MAXBN$, being composed of similar $\triangle$ similarly placed, are similar.

$$\frac{M'N'}{MN} = \frac{OE}{OD}. \quad (?) \qquad \text{Whence } \frac{S}{s} = \frac{\overline{OE}^2}{\overline{OD}^2}. \quad (?)$$

$$\frac{S-s}{S} = \frac{\overline{OE}^2 - \overline{OD}^2}{\overline{OE}^2} \text{ and } S - s = \frac{S}{\overline{OE}^2}(\overline{OE}^2 - \overline{OD}^2). \quad (?)$$

Show that $S - s$ can be made as small as we please, but not equal to zero.

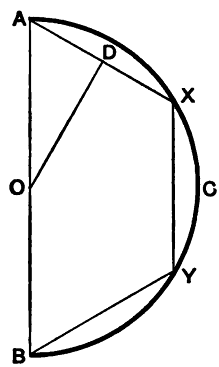
Show that $S - S'$ and $S' - s$ are each less than $S - s$.

Show that S' is the limit of S and of s. Q.E.D

1183. Corollary. *If a semicircle with a regular inscribed or circumscribed semipolygon be revolved about its diameter as an axis, and if the number of sides of the polygon be indefinitely increased, the surface of the sphere generated by the semicircumference is the limit of the surface generated by the perimeter of the regular semipolygon.*

Proposition XXVIII. Theorem

1184. *The surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*



Let the semicircle ACB, with the regular inscribed semipolygon $AXYB$, be revolved about the diameter AB as an axis.

To Prove that the surface of the sphere generated by $ACB = AB \times 2\pi OA$.

Proof. Surface $AXYB = AB \times 2\pi OD$.

Let the number of sides of $AXYB$ be indefinitely increased.

The limit of the variable surface $AXYB$ is the surface ACB (?), and the limit of OD is OA. (?)

Show that the surface of the sphere $= AB \times 2\pi OA$. Q.E.D.

1185. Corollary I. *The surface of a sphere is four times the area of its great circle.*

1186. Corollary II. *The area of a zone is equal to the product of its altitude by the circumference of a great circle of the sphere.*

1187. Corollary III. *The surfaces of spheres are to each other as the squares of their radii, or as the squares of their diameters.*

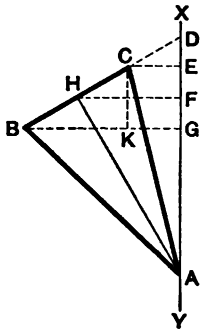
1188. Exercise. The radius of a sphere is 9 in. Find its surface.

1189. Exercise. Find the radius of a sphere if the area of its trirectangular triangle is 1413.72 sq. ft.

1190. Exercise. If it costs a dollars to gild a sphere whose radius is b in., what will it cost to gild one whose radius is c in.?

Proposition XXIX. Theorem

1191. *If a line be drawn in the plane of an isosceles triangle through its vertex and not intersecting the triangle, and if the triangle be revolved about this line as an axis, the volume generated by the triangle will be equal to the product of one third of its altitude by the surface generated by its base.*



Case I. When the prolonged base of the isosceles △ meets the axis.

Let the isosceles △ ABC be revolved about XY as an axis.

To Prove

$$\text{vol. } ABC = \text{surface } BC \times \tfrac{1}{3} AH.$$

Proof. Draw CE, HF, and BG ⊥ to XY, and CK ⊥ to BG.

$$\text{Vol. } ABC = \text{vol. } ABD - \text{vol. } ACD.$$

$$\text{Vol. } ABC = \tfrac{1}{3} AD\pi\overline{BG}^2 - \tfrac{1}{3} AD\pi\overline{CE}^2. \quad (\S\ 1041.)$$

$$\text{Vol. } ABC = \tfrac{1}{3}\pi AD(\overline{BG}^2 - \overline{CE}^2)$$

$$= \tfrac{1}{3}\pi AD(BG - CE)(BG + CE)$$

$$= \tfrac{1}{3}\pi AD \cdot BK \cdot 2\,HF. \quad (?) \qquad (1)$$

Since △ AHD, AHF, and CKB are similar (?)

$$\frac{AD}{HA} = \frac{HA}{AF}. \quad (2) \qquad \frac{BK}{CK} = \frac{AF}{HF}. \quad (3)$$

Multiplying (2) by (3) and clearing of fractions,

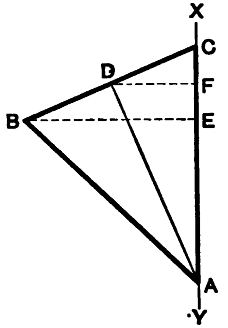
$$AD \cdot BK \cdot HF = \overline{HA}^2 \cdot CK. \qquad (4)$$

Substitute (4) in (1).

$$\text{Vol. } ABC = \tfrac{2}{3}\,\pi\overline{HA}^2 \cdot CK = EG \cdot 2\,\pi AH \cdot \tfrac{1}{3}\,AH$$

$$= \text{surface } BC \times \tfrac{1}{3}\,AH. \qquad \text{Q.E.D.}$$

CASE II. When the axis coincides with one of the equal sides of the isosceles triangle.



Let the isosceles $\triangle ABC$ be revolved about XY as an axis.

To Prove

vol. $ABC =$ surface $BC \times \frac{1}{3} AD$.

Proof.

Vol. $ABC = \frac{1}{3} AC\pi\overline{BE}^2$. (?) (1)

$\triangle$s ADC, DCF, and BCE are similar. (?)

$\frac{AC}{AD} = \frac{BC}{BE}$. (2) $\qquad \frac{BE}{CE} = \frac{AD}{DC}$. (3)

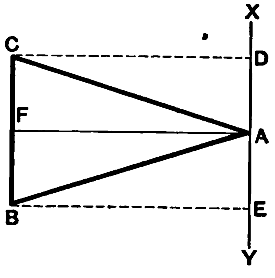
Multiplying (2) by (3) and clearing of fractions,

$$AC \cdot \overline{BE}^2 = 2\,CE \cdot \overline{AD}^2. \quad (?) \qquad (4)$$

Substitute (4) in (1).

Show that vol. $ABC =$ surface $BC \times \frac{1}{3} AD$. Q.E.D.

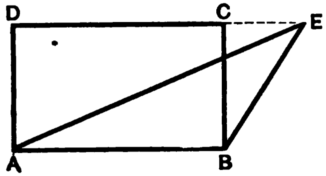
CASE III. When the base of the isosceles triangle is parallel to the axis.



Vol. $ABC =$ vol. $DCBE -$ vol. $ADC -$ vol. ABE.

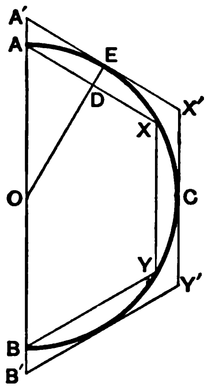
[Proof is left to the student.]

1192. Exercise. If the rectangle *ABCD* and the triangle *ABE* have a common base and equal altitudes, and if they be revolved about the common base as an axis, the volume generated by the rectangle will be three times the volume generated by the triangle.



Proposition XXX. Theorem

1193. *If a semicircle with a regular inscribed or circumscribed semipolygon be revolved about its diameter as an axis, and if the number of sides of the semipolygon be indefinitely increased, the volume of the sphere generated by the semicircle is the limit of the volume generated by the regular semipolygon.*



Let *ACB* be the semicircle, *AXYB* and *A'X'Y'B'* be the regular semipolygons, and let the number of their sides be indefinitely increased.

To Prove volume *ACB* is the limit of volume *AXYB* and of volume *A'X'Y'B'*.

Proof. Designate volumes $A'X'Y'B'$, $AXYB$, and ACB by V, v, and V' respectively; and surfaces $A'X'Y'B'$, $AXYB$, and ACB by S, s, and S' respectively.

$$\frac{V}{v}=\frac{S \times \frac{1}{3} OE}{s \times \frac{1}{3} OD}. \quad (?)$$

Show that $$\frac{S}{s}=\frac{\overline{OE}^2}{\overline{OD}^2}.$$ (See proof of § 1182.)

$$\frac{V}{v}=\frac{\overline{OE}^3}{\overline{OD}^3}. \quad (?) \qquad \frac{V-v}{V}=\frac{\overline{OE}^3-\overline{OD}^3}{\overline{OE}^3}. \quad (?)$$

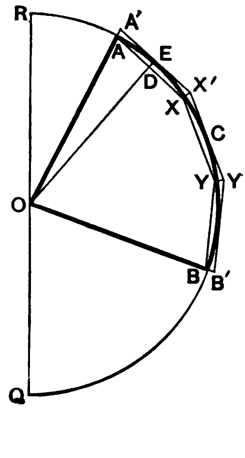
$$V-v=\frac{V}{\overline{OE}^3}(\overline{OE}^3-\overline{OD}^3).$$

Show that $V-v$ can be made as small as we please, but not equal to zero.

$$V'<V \text{ and } V'>v. \quad (?)$$

Show that $V-V'$ and $V'-v$ can each be made as small as we please, but not equal to zero.

Show that V' is the limit of V or v. Q.E.D.

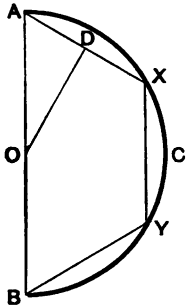


1194. Corollary. *If the sector $OACB$ together with the polygons $OAXYB$ and $OA'X'Y'B'$ be revolved about the diameter RQ as an axis, and if the number of divisions of the regular broken lines $AXYB$ and $A'X'Y'B'$ be indefinitely increased, the spherical sector generated by $OACB$ is the limit of the volumes generated by $OAXYB$ and $OA'X'Y'B'$.*

[Proof similar to that of § 1193.]

Proposition XXXI. Theorem

1195. *The volume of a sphere is equal to the product of its surface by one third of its radius.*



Let the semicircle ACB, with the regular inscribed semipolygon $AXYB$, be revolved about the diameter AB as an axis.

To Prove that the volume of the sphere generated by $ACB =$ its surface $\times \frac{1}{3}$ of its radius.

Proof.

Vol. $AXYB =$ surface $AXYB \times \frac{1}{3} OD$. (?)

Let the number of sides of $AXYB$ be indefinitely increased.

The limit of the variable volume $AXYB$ is the volume of the sphere (?); the limit of surface $AXYB$ is the surface of the sphere (?); and the limit of OD is the radius of the sphere. (?)

Show that vol. $ACB =$ surface $ACB \times \frac{1}{3} OA$. Q.E.D.

1196. Corollary I. *The volume of a sphere* $= \frac{4}{3}\pi R^3$ *or* $\frac{1}{6}\pi D^3$.

1197. Corollary II. *The volume of a spherical sector is equal to the product of the zone that forms its base by one third the radius of the sphere.* [Proof similar to that of § 1195.]

1198. Corollary III. *The volume of a spherical pyramid is equal to the product of its base by one third the radius of the sphere.*

[By § 1172, $\dfrac{\text{vol. spher. pyramid}}{\text{vol. trirectangular pyramid}} = \dfrac{\text{base of spher. pyr.}}{\text{trirectangular } \triangle}$.

$$\therefore \frac{\text{vol. spher. pyramid}}{\frac{1}{8}\text{ vol. sphere}} = \frac{\text{base of spher. pyr.}}{\frac{1}{8}\text{ surface sphere}}.$$

Whence vol. spher. pyramid $=$ base $\times \frac{1}{3}$ radius.]

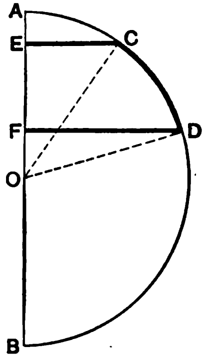
1199. Corollary IV. *The volumes of spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

1200. Exercise. The number of cubic feet in the volume of a certain sphere is the same as the number of square feet in its surface. Find its diameter.

Proposition XXXII. Theorem

1201. *The volume of a spherical segment is equal to the product of one half the sum of its bases by its altitude, increased by the volume of a sphere having that altitude for its diameter.*

Let V designate the volume of the spherical segment generated by revolving $ECDF$ about AB as an axis, R the radius of the sphere, a the altitude of the zone, r_1 and r_2 the radii of the bases generated by EC and FD respectively, and m the distance FO.



To Prove $V = \dfrac{a(\pi r_1^2 + \pi r_2^2)}{2} + \frac{1}{6}\pi a^3.$

Proof.

$V =$ vol. OCD + vol. OCE − vol. ODF.

$V = \frac{2}{3}\pi a R^2 + \frac{1}{3}\pi(a+m)r_1^2 - \frac{1}{3}\pi m r_2^2.$ (?) (1)

$R^2 = r_1^2 + (a+m)^2.$ (2)

$R^2 = r_2^2 + m^2.$ (3)

Equate (2) and (3), whence $m = \dfrac{r_2^2 - r_1^2 - a^2}{2a}.$ (4)

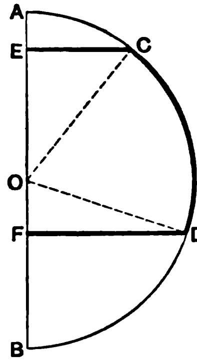
Substitute (4) in (3),

$$R^2 = \frac{a^4 + r_1^4 + r_2^4 + 2r_1^2a^2 + 2r_2^2a^2 - 2r_1^2r_2^2}{4a^2}. \quad (5)$$

Substitute (4) and (5) in (1), and show

$$V = \frac{a(\pi r_1^2 + \pi r_2^2)}{2} + \frac{1}{6}\pi a^3. \qquad \text{Q.E.D.}$$

1202. Corollary I. *If the center of the sphere falls within the spherical segment, then*



$$V = \frac{2}{3}\pi a R^2 + \frac{1}{3}\pi(a - m)r_1^2 + \frac{1}{3}\pi m r_2^2, \quad (1)$$

and
$$m = \frac{a^2 + r_1^2 - r_2^2}{2a}. \quad (4)$$

The value of R^2 in (5) is unchanged, as is the final result

$$V = \frac{a(\pi r_1^2 + \pi r_2^2)}{2} + \frac{1}{6}\pi a^3. \quad (6)$$

1203. Corollary II. *The volume of a spherical segment of one base is equal to the product of one half its base by its altitude, increased by the volume of a sphere having that altitude for its diameter.* [This may be derived by letting $r_1 = 0$ in (6).]

1204. Exercise. The radii of the bases of a spherical segment are 18 in. and 24 in. respectively. The radius of the sphere is 30 in. Find the volume of the segment.

EXERCISES

1. The volume of a sphere is to the volume of a circumscribed cube as π is to 6.

2. Describe a spherical surface with a given radius that shall pass through three given points.

3. Describe a spherical surface with a given radius that shall pass through two given points and be tangent to a given plane.

4. Through a given point within a sphere pass a plane such that the area of the section formed shall be a minimum. A maximum.

5. Find the altitude of a zone whose area is equal to that of a great circle of the sphere.

6. The surface of a sphere is to the total surface of a circumscribed cylinder as 2 is to 3.

7. Describe a spherical surface with a given radius passing through two given points and tangent to a given line.

8. Describe a spherical surface with a given radius passing through two given points and tangent to a given sphere.

9. Circumscribe a circle about a spherical triangle, and inscribe a circle in it.

10. The volume of a sphere is 11309.76 cu. in. Find its surface.

11. The angles of a spherical triangle are 60°, 80°, and 100° respectively, and the radius of the sphere is 15 in. Find the volume of the spherical pyramid of which this triangle is the base.

12. Describe a spherical surface with a given radius passing through a given point and tangent to two given lines.

13. Describe a spherical surface with a given radius passing through a given point and tangent to two given spheres.

14. Describe a spherical surface with a given radius passing through a given point and tangent to two given planes.

15. A cone of revolution whose slant height is equal to the diameter of its base is circumscribed about a sphere. If the radius of the sphere is a in., what is the volume of the cone?

16. Find the area of a spherical quadrilateral whose angles are 117°, 129°, 142°, and 154°, on a sphere whose volume is 2304.

17. Determine the locus of points equally distant from three planes, each of which is perpendicular to the other two.

18. A point P is at a distance of 30 ft. from the center of a sphere whose radius is 10 ft. A right circular cone circumscribing the sphere has its vertex at P and its base tangent to the sphere. Find the volume of the cone.

19. If h is the height of an aeronaut and r the radius of the earth, the extent of the surface visible to the aeronaut is $\frac{2\pi r^2 h}{r+h}$.

20. What portion of the earth's surface is visible from a point whose distance above the surface of the earth is equal to the earth's radius?

21. Find the volume of a spherical segment if the diameter of each base is 8 ft., and the altitude is 6 ft.

22. The volumes of polyhedrons circumscribed about the same sphere are to each other as their surfaces.

23. Describe a spherical surface with a given radius passing through a given point and tangent to a given line and also to a given sphere.

24. Describe a spherical surface with a given radius passing through a given point and tangent to a given line and also to a given plane.

25. Find the volume of spherical wedge whose angle is 7° 30′ if the radius of the sphere is 100 in.

26. How many degrees in the polar distance of a circle whose plane is $10\sqrt{2}$ in. from the center of the sphere whose diameter is 40 in.?

27. Describe a spherical surface with a given radius tangent to two given lines and a sphere.

28. Describe a spherical surface with a given radius tangent to two given lines and a plane.

29. The volume of a sphere is to the volume of a circumscribed cylinder as 2 is to 3.

30. If the distance between the centers of two intersecting spheres whose radii are a ft. and b ft. respectively is c ft., find the diameter of their circle of intersection.

31. If one side of a spherical triangle is a quadrant and an angle adjacent to the side is acute, the side opposite this angle is less than 90°.

32. An edge of a regular tetrahedron is 10 in. Find the radius of the inscribed sphere and of the circumscribed sphere.

33. The area of a triangle is 57 sq. in. Two of its angles are 76° and 85° respectively. Find the third angle if the surface of the sphere contains 216 sq. in.

34. The sides of a spherical triangle are 65°, 75°, and 90°, and the surface of the sphere is 500 sq. ft. Find the area of its polar triangle.

35. The area of a spherical hexagon is one eighth of the surface of the sphere. Five of its angles are 150°, 120°, 90°, 130°, and 70°. Find the remaining angle.